

PRELIMINARIES (EGA 0)

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§1. RINGS OF FRACTIONS

1.0. Rings and Algebras

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(1.0.1). All the rings considered in this treatise will have a *unit element*; all the modules over such a ring will be assumed to be *unitary*; the ring homomorphisms will always be assumed to *send the unit element to the unit element*; unless otherwise stated, a subring of a ring A will be assumed to *contain the unit element of A* . We will focus in particular on *commutative* rings, and when we speak of a ring without specifying any details, it will be implied that it is commutative. If A is a not-necessarily-commutative ring, by A -module we will mean a *left* module unless stated otherwise.

(1.0.2). Let A and B be not-necessarily-commutative rings and $\phi : A \rightarrow B$ a homomorphism. Any left (resp. right) B -module M can be provided with a left (resp. right) A -module structure by $a \cdot m = \phi(a) \cdot m$ (resp. $m \cdot a = m \cdot \phi(a)$); when it will be necessary to distinguish M as an A -module or a B -module, we will denote by $M_{[\phi]}$ the left (resp. right) A -module defined as such. If L is an A -module, then a homomorphism $u : L \rightarrow M_{[\phi]}$ is a homomorphism of abelian groups such that $u(a \cdot x) = \phi(a) \cdot u(x)$ for $a \in A, x \in L$; we will also say that it is a ϕ -homomorphism $L \rightarrow M$, and

that the pair (ϕ, u) (or, by abuse of language, u) is a *di-homomorphism* from (A, L) to (B, M) . The pairs (A, L) consisting of a ring A and an A -module L thus form a *category* whose morphisms are di-homomorphisms.

(1.0.3). Under the hypotheses of (1.0.2), if \mathfrak{J} is a left (resp. right) ideal of A , we denote by $B\mathfrak{J}$ (resp. $\mathfrak{J}B$) the left (resp. right) ideal $B\phi(\mathfrak{J})$ (resp. $\phi(\mathfrak{J})B$) of B generated by $\phi(\mathfrak{J})$; it is also the image of the canonical homomorphism $B \otimes_A \mathfrak{J} \rightarrow B$ (resp. $\mathfrak{J} \otimes_A B \rightarrow B$) of left (resp. right) B -modules.

(1.0.4). If A is a (commutative) ring, and B a not-necessarily-commutative ring, then the data of a structure of an A -algebra on B is equivalent to the data of a ring homomorphism $\phi : A \rightarrow B$ such that $\phi(A)$ is contained in the center of B . For all ideals \mathfrak{J} of A , $\mathfrak{J}B = B\mathfrak{J}$ is then a two-sided ideal of B , and for every B -module M , $\mathfrak{J}M$ is then a B -module equal to $(B\mathfrak{J})M$.

(1.0.5). We will not dwell much on the notions of *modules of finite type* and (commutative) *algebras of finite type*; to say that an A -module M is of finite type means that there exists an exact sequence $A^p \rightarrow M \rightarrow 0$. We say that an A -module M admits a *finite presentation* if it is isomorphic to the cokernel of a homomorphism $A^p \rightarrow A^q$, or, in other words, if there exists an exact sequence $A^p \rightarrow A^q \rightarrow M \rightarrow 0$. We note that for a *Noetherian* ring A , every A -module of finite type admits a finite presentation.

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Let us recall that an A -algebra B is said to be *integral* over A if every element in B is a root in B of a monic polynomial with coefficients in A ; equivalently, if every element of B is contained in a subalgebra of B which is an A -module of finite type. When this is so, and if B is commutative, the subalgebra of B generated by a finite subset of B is an A -module of finite type; for a commutative algebra B to be integral and of finite type over A , it is necessary and sufficient that B be an A -module of finite type; we also say that B is an *integral A -algebra of finite type* (or simply *finite*, if there is no chance of confusion). It should be noted that in these definitions, it is not assumed that the homomorphism $A \rightarrow B$ defining the A -algebra structure is injective.

(1.0.6). An *integral ring* (or an *integral domain*) is a ring in which the product of a finite family of elements $\neq 0$ is $\neq 0$; equivalently, in such a ring, we have $0 \neq 1$, and the product of two elements $\neq 0$ is $\neq 0$. A *prime ideal* of a ring A is an ideal \mathfrak{p} such that A/\mathfrak{p} is integral; this implies that $\mathfrak{p} \neq A$. For a ring A to have at least one prime ideal, it is necessary and sufficient that $A \neq \{0\}$.

(1.0.7). A *local ring* is a ring A in which there exists a unique maximal ideal, which is thus the complement of the invertible elements, and contains all the ideals $\neq A$. If A and B are local rings, and \mathfrak{m} and \mathfrak{n} their respective maximal ideals, then we say that a homomorphism $\phi : A \rightarrow B$ is *local* if $\phi(\mathfrak{m}) \subset \mathfrak{n}$ (or, equivalently, if $\phi^{-1}(\mathfrak{n}) = \mathfrak{m}$). By passing to quotients, such a homomorphism then defines a monomorphism from the residue field A/\mathfrak{m} to the residue field B/\mathfrak{n} . The composition of any two local homomorphisms is a local homomorphism.

1.1. Radical of an ideal. Nilradical and radical of a ring

(1.1.1). Let \mathfrak{a} be an ideal of a ring A ; the *radical* of \mathfrak{a} , denoted by $\tau(\mathfrak{a})$, is the set of $x \in A$ such that $x^n \in \mathfrak{a}$ for an integer $n > 0$; it is an ideal containing \mathfrak{a} . We have $\tau(\tau(\mathfrak{a})) = \tau(\mathfrak{a})$; the relation $\mathfrak{a} \subset \mathfrak{b}$ implies $\tau(\mathfrak{a}) \subset \tau(\mathfrak{b})$; the radical of a finite intersection of ideals is the intersection of their radicals. If ϕ is a homomorphism from another ring A' to A , then we have $\tau(\phi^{-1}(\mathfrak{a})) = \phi^{-1}(\tau(\mathfrak{a}))$ for any ideal $\mathfrak{a} \subset A$. For an ideal to be the radical of an ideal, it is necessary and sufficient that it be an intersection of prime ideals. The radical of an ideal \mathfrak{a} is the intersection of the *minimal* prime ideals which contain \mathfrak{a} ; if A is Noetherian, there are finitely many of these minimal prime ideals.

The radical of the ideal (0) is also called the *nilradical* of A ; it is the set \mathfrak{N} of the nilpotent elements of A . We say that the ring A is *reduced* if $\mathfrak{N} = (0)$; for every ring A , the quotient A/\mathfrak{N} of A by its nilradical is a reduced ring.

(1.1.2). Recall that the *radical* $\tau(A)$ of a (not-necessarily-commutative) ring A is the intersection of the maximal left ideals of A (and also the intersection of maximal right ideals). The radical of $A/\tau(A)$ is (0) .

1.2. Modules and rings of fractions

(1.2.1). We say that a subset S of a ring A is *multiplicative* if $1 \in S$ and the product of two elements of S is in S . The examples which will be the most important in what follows are: 1st, the set S_f of powers f^n ($n \geq 0$) of an element $f \in A$; and 2nd, the complement $A - \mathfrak{p}$ of a *prime* ideal \mathfrak{p} of A .

(1.2.2). Let S be a multiplicative subset of a ring A , and M an A -module; on the set $M \times S$, the relation between pairs (m_1, s_1) and (m_2, s_2) :

$$\text{“there exists an } s \in S \text{ such that } s(s_1m_2 - s_2m_1) = 0\text{”}$$

is an equivalence relation. We denote by $S^{-1}M$ the quotient set of $M \times S$ by this relation, and by m/s the canonical image of the pair (m, s) in $S^{-1}M$; we call $i_M^S : m \mapsto m/1$ (also denoted i^S) the *canonical* map from M to $S^{-1}M$. This map is, in general, neither injective nor surjective; its kernel is the set of $m \in M$ such that there exists an $s \in S$ for which $sm = 0$.

On $S^{-1}M$ we define an additive group law by setting

$$(m_1/s_1) + (m_2/s_2) = (s_2m_1 + s_1m_2)/(s_1s_2)$$

(one can check that it is independent of the choice of representative of the elements of $S^{-1}M$, which are equivalence classes). On $S^{-1}A$ we further define a multiplicative law by setting $(a_1/s_1)(a_2/s_2) = (a_1a_2)/(s_1s_2)$, and finally an exterior law on $S^{-1}M$, acted on by the set of elements of $S^{-1}A$, by setting $(a/s)(m/s') = (am)/(ss')$. It can then be shown that $S^{-1}A$ is endowed with a ring structure (called *the ring of fractions of A with denominators in S*) and $S^{-1}M$ with the structure of an $S^{-1}A$ -module (called *the module of fractions of M with denominators in S*); for all $s \in S$, $s/1$ is invertible in $S^{-1}A$, its inverse being $1/s$. The canonical map i_A^S (resp. i_M^S) is a ring homomorphism (resp. a homomorphism of A -modules, $S^{-1}M$ being considered as an A -module by means of the homomorphism $i_A^S : A \rightarrow S^{-1}A$).

(1.2.3). If $S_f = \{f^n\}_{n \geq 0}$ for a $f \in A$, we write A_f and M_f instead of $S_f^{-1}A$ and $S_f^{-1}M$; when A_f is considered as algebra over A , we can write $A_f = A[1/f]$. A_f is isomorphic to the quotient algebra $A[T]/(fT - 1)A[T]$. When $f = 1$, A_f and M_f are canonically identified with A and M ; if f is nilpotent, then A_f and M_f are 0.

When $S = A - \mathfrak{p}$, with \mathfrak{p} a prime ideal of A , we write $A_{\mathfrak{p}}$ and $M_{\mathfrak{p}}$ instead of $S^{-1}A$ and $S^{-1}M$; $A_{\mathfrak{p}}$ is a *local ring* whose maximal ideal \mathfrak{q} is generated by $i_A^S(\mathfrak{p})$, and we have $(i_A^S)^{-1}(\mathfrak{q}) = \mathfrak{p}$; by passing to quotients, i_A^S gives a monomorphism from the integral ring A/\mathfrak{p} to the field $A_{\mathfrak{p}}/\mathfrak{q}$, which can be identified with the field of fractions of A/\mathfrak{p} .

(1.2.4). The ring of fractions $S^{-1}A$ and the canonical homomorphism i_A^S are a solution to a *universal mapping problem*: any homomorphism u from A to a ring B such that $u(S)$ is composed of *invertible* elements in B factors uniquely as

$$u : A \xrightarrow{i_A^S} S^{-1}A \xrightarrow{u^*} B$$

where u^* is a ring homomorphism. Under the same hypotheses, let M be an A -module, N a B -module, and $v : M \rightarrow N$ a homomorphism of A -modules (for the B -module structure on N defined by $u : A \rightarrow B$); then v factors uniquely as

$$v : M \xrightarrow{i_M^S} S^{-1}M \xrightarrow{v^*} N$$

where v^* is a homomorphism of $S^{-1}A$ -modules (for the $S^{-1}A$ -module structure on N defined by u^*).

(1.2.5). We define a canonical isomorphism $S^{-1}A \otimes_A M \simeq S^{-1}M$ of $S^{-1}A$ -modules, sending the element $(a/s) \otimes m$ to the element $(am)/s$, with the inverse isomorphism sending m/s to $(1/s) \otimes m$.

(1.2.6). For every ideal \mathfrak{a}' of $S^{-1}A$, $\mathfrak{a} = (i_A^S)^{-1}(\mathfrak{a}')$ is an ideal of A , and \mathfrak{a}' is the ideal of $S^{-1}A$ generated by $i_A^S(\mathfrak{a})$, which can be identified with $S^{-1}\mathfrak{a}$ (1.3.2). The map $\mathfrak{p}' \mapsto (i_A^S)^{-1}(\mathfrak{p}')$ is an isomorphism, for the structure given by ordering, from the set of *prime* ideals of $S^{-1}A$ to the set of prime ideals \mathfrak{p} of A such that $\mathfrak{p} \cap S = \emptyset$. In addition, the local rings $A_{\mathfrak{p}}$ and $(S^{-1}A)_{S^{-1}\mathfrak{p}}$ are canonically isomorphic (1.5.1).

(1.2.7). When A is an *integral ring*, for which K denotes its field of fractions, the canonical map $i_A^S : A \rightarrow S^{-1}A$ is injective for any multiplicative subset S not containing 0, and $S^{-1}A$ is then canonically identified with a subring of K containing A . In particular, for every prime ideal \mathfrak{p} of A , $A_{\mathfrak{p}}$ is a local ring containing A , with maximal ideal $\mathfrak{p}A_{\mathfrak{p}}$, and we have $\mathfrak{p}A_{\mathfrak{p}} \cap A = \mathfrak{p}$.

(1.2.8). If A is a *reduced ring* (1.1.1), so is $S^{-1}A$: indeed, if $(x/s)^n = 0$ for $x \in A, s \in S$, then this means that there exists an $s' \in S$ such that $s'x^n = 0$, hence $(s'x)^n = 0$, which, by hypothesis, implies $s'x = 0$, so $x/s = 0$.

1.3. Functorial properties

(1.3.1). Let M and N be A -modules, and u an A -homomorphism $M \rightarrow N$. If S is a multiplicative subset of A , we define a $S^{-1}A$ -homomorphism $S^{-1}M \rightarrow S^{-1}N$, denoted by $S^{-1}u$, by setting $S^{-1}u(m/s) = u(m)/s$; if $S^{-1}M$ and $S^{-1}N$ are canonically identified with $S^{-1}A \otimes_A M$ and $S^{-1}A \otimes_A N$ (1.2.5), then $S^{-1}u$ is considered as $1 \otimes u$. If P is a third A -module, and v an A -homomorphism $N \rightarrow P$, we have $S^{-1}(v \circ u) = (S^{-1}v) \circ (S^{-1}u)$; in other words, $S^{-1}M$ is a *covariant functor* in M , from the category of A -modules to that of $S^{-1}A$ -modules (A and S being fixed).

(1.3.2). The functor $S^{-1}M$ is *exact*; in other words, if the sequence

$$M \xrightarrow{u} N \xrightarrow{v} P$$

is exact, then so is the sequence

$$S^{-1}M \xrightarrow{S^{-1}u} S^{-1}N \xrightarrow{S^{-1}v} S^{-1}P.$$

In particular, if $u : M \rightarrow N$ is injective (resp. surjective), the same is true for $S^{-1}u$; if N and P are submodules of M , $S^{-1}N$ and $S^{-1}P$ are canonically identified with submodules of $S^{-1}M$, and we have

$$S^{-1}(N + P) = S^{-1}N + S^{-1}P \text{ and } S^{-1}(N \cap P) = (S^{-1}N) \cap (S^{-1}P).$$

(1.3.3). Let $(M_\alpha, \phi_{\beta\alpha})$ be an inductive system of A -modules; then $(S^{-1}M_\alpha, S^{-1}\phi_{\beta\alpha})$ is an inductive system of $S^{-1}A$ -modules. Expressing the $S^{-1}M_\alpha$ and $S^{-1}\phi_{\beta\alpha}$ as tensor products ((1.2.5) and (1.3.1)), it follows from the permutability of the tensor product and inductive limit operations that we have a canonical isomorphism

$$S^{-1} \varinjlim M_\alpha \simeq \varinjlim S^{-1}M_\alpha$$

which we can further express by saying that the functor $S^{-1}M$ (in M) *commutes with inductive limits*.

(1.3.4). Let M and N be A -modules; there is a canonical *functorial* (in M and N) isomorphism

$$(S^{-1}M) \otimes_{S^{-1}A} (S^{-1}N) \simeq S^{-1}(M \otimes_A N)$$

which sends $(m/s) \otimes (n/t)$ to $(m \otimes n)/st$.

(1.3.5). We also have a *functorial* (in M and N) homomorphism

$$S^{-1} \text{Hom}_A(M, N) \longrightarrow \text{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N)$$

which sends u/s to the homomorphism $m/t \mapsto u(m)/st$. When M has a finite presentation, the above homomorphism is an *isomorphism*: it is immediate when M is of the form A^r , and we pass to the general case by starting with the exact sequence $A^p \rightarrow A^q \rightarrow M \rightarrow 0$ and using the exactness of the functor $S^{-1}M$ and the left-exactness of the functor $\text{Hom}_A(M, N)$ in M . Note that this is always the case when A is *Noetherian* and the A -module M is of *finite type*.

1.4. Change of multiplicative subset

(1.4.1). Let S and T be multiplicative subsets of a ring A such that $S \subset T$; there exists a canonical homomorphism $\rho_A^{T,S}$ (or simply $\rho^{T,S}$) from $S^{-1}A$ to $T^{-1}A$, sending the element denoted a/s of $S^{-1}A$ to the element denoted a/s in $T^{-1}A$; we have $i_A^T = \rho_A^{T,S} \circ i_A^S$. For every A -module M , there exists, in the same way, an $S^{-1}A$ -linear map from $S^{-1}M$ to $T^{-1}M$ (the latter considered as an $S^{-1}A$ -module by the homomorphism $\rho_A^{T,S}$), which sends the element m/s of $S^{-1}M$ to the element m/s of $T^{-1}M$; we denote this map by $\rho_M^{T,S}$, or simply $\rho^{T,S}$, and we still have $i_M^T = \rho_M^{T,S} \circ i_M^S$; by the canonical identification (1.2.5), $\rho_M^{T,S}$ is identified with $\rho_A^{T,S} \otimes 1$. The homomorphism $\rho_M^{T,S}$ is a *functorial morphism* (or natural transformation) from the functor $S^{-1}M$ to the functor $T^{-1}M$, in other words, the diagram

$$\begin{array}{ccc} S^{-1}M & \xrightarrow{S^{-1}u} & S^{-1}N \\ \rho_M^{T,S} \downarrow & & \downarrow \rho_N^{T,S} \\ T^{-1}M & \xrightarrow{T^{-1}u} & T^{-1}N \end{array}$$

is commutative, for every homomorphism $u : M \rightarrow N$; $T^{-1}u$ is entirely determined by $S^{-1}u$, since, for $m \in M$ and $t \in T$, we have

$$(T^{-1}u)(m/t) = (t/1)^{-1} \rho^{T,S}((S^{-1}u)(m/1)).$$

(1.4.2). With the same notation, for A -modules M and N , the diagrams (cf. (1.3.4) and (1.3.5))

$$\begin{array}{ccc} (S^{-1}M) \otimes_{S^{-1}A} (S^{-1}N) & \xrightarrow{\sim} & S^{-1}(M \otimes_A N) \\ \downarrow & & \downarrow \\ (T^{-1}M) \otimes_{T^{-1}A} (T^{-1}N) & \xrightarrow{\sim} & T^{-1}(M \otimes_A N) \end{array}$$

$$\begin{array}{ccc} S^{-1} \operatorname{Hom}_A(M, N) & \longrightarrow & \operatorname{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N) \\ \downarrow & & \downarrow \\ T^{-1} \operatorname{Hom}_A(M, N) & \longrightarrow & \operatorname{Hom}_{T^{-1}A}(T^{-1}M, T^{-1}N) \end{array}$$

are commutative.

(1.4.3). There is an important case, in which the homomorphism $\rho^{T,S}$ is *bijective*, when we then know that every element of T is a divisor of an element of S ; we then identify the modules $S^{-1}M$ and $T^{-1}M$ via $\rho^{T,S}$. We say that S is *saturated* if every divisor in A of an element of S is in S ; by replacing S with the set T of all the divisors of the elements of S (a set which is multiplicative and saturated), we see that we can always, if we wish, consider only modules of fractions $S^{-1}M$, where S is saturated.

(1.4.4). If S , T , and U are three multiplicative subsets of A such that $S \subset T \subset U$, then we have

$$\rho^{U,S} = \rho^{U,T} \circ \rho^{T,S}.$$

(1.4.5). Consider an *increasing filtered family* (S_α) of multiplicative subsets of A (we write $\alpha \leq \beta$ for $S_\alpha \subset S_\beta$), and let S be the multiplicative subset $\bigcup_\alpha S_\alpha$; let us put $\rho_{\beta\alpha} = \rho_A^{S_\beta, S_\alpha}$ for $\alpha \leq \beta$; according to (1.4.4), the homomorphisms $\rho_{\beta\alpha}$ define a ring A' as the *inductive limit* of the inductive system of rings $(S_\alpha^{-1}A, \rho_{\beta\alpha})$. Let ρ_α be the canonical map $S_\alpha^{-1}A \rightarrow A'$, and let $\phi_\alpha = \rho_A^{S, S_\alpha}$; as $\phi_\alpha = \phi_\beta \circ \rho_{\beta\alpha}$ for $\alpha \leq \beta$

according to (1.4.4), we can uniquely define a homomorphism $\phi : A' \rightarrow S^{-1}A$ such that the diagram

$$\begin{array}{ccc}
 & S_{\alpha}^{-1}A & \\
 \rho_{\alpha} \swarrow & \downarrow \rho_{\beta\alpha} & \searrow \phi_{\alpha} \\
 & S_{\beta}^{-1}A & \\
 \rho_{\beta} \swarrow & & \searrow \phi_{\beta} \\
 A' & \xrightarrow{\phi} & S^{-1}A
 \end{array} \quad (\alpha \leq \beta)$$

is commutative. In fact, ϕ is an *isomorphism*; it is indeed immediate by construction that ϕ is surjective. On the other hand, if $\rho_{\alpha}(a/s_{\alpha}) \in A'$ is such that $\phi(\rho_{\alpha}(a/s_{\alpha})) = 0$, then this means that $a/s_{\alpha} = 0$ in $S^{-1}A$, that is to say that there exists an $s \in S$ such that $sa = 0$; but there is a $\beta \geq \alpha$ such that $s \in S_{\beta}$, and consequently, as $\rho_{\alpha}(a/s_{\alpha}) = \rho_{\beta}(sa/ss_{\alpha}) = 0$, we find that ϕ is injective. The case for an A -module M is treated likewise, and we have thus defined canonical isomorphisms

$$\varinjlim S_{\alpha}^{-1}A \simeq (\varinjlim S_{\alpha})^{-1}A, \quad \varinjlim S_{\alpha}^{-1}M \simeq (\varinjlim S_{\alpha})^{-1}M,$$

the second being *functorial* in M .

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(1.4.6). Let S_1 and S_2 be multiplicative subsets of A ; then S_1S_2 is also a multiplicative subset of A . Let us denote by S_2' the canonical image of S_2 in the ring $S_1^{-1}A$, which is a multiplicative subset of this ring. For every A -module M there is then a functorial isomorphism

$$S_2'^{-1}(S_1^{-1}M) \simeq (S_1S_2)^{-1}M$$

which sends $(m/s_1)/(s_2/1)$ to the element $m/(s_1s_2)$.

1.5. Change of ring

(1.5.1). Let A and A' be rings, ϕ a homomorphism $A' \rightarrow A$, and S (resp. S') a multiplicative subset of A (resp. A'), such that $\phi(S') \subset S$; the composition homomorphism $A' \xrightarrow{\phi} A \rightarrow S^{-1}A$ factors as

$$A' \longrightarrow S'^{-1}A' \xrightarrow{\phi^{S'}} S^{-1}A,$$

by (1.2.4); where $\phi^{S'}(a'/s') = \phi(a')/\phi(s')$. If $A = \phi(A')$ and $S = \phi(S')$, then $\phi^{S'}$ is *surjective*. If $A' = A$ and ϕ is the identity, then $\phi^{S'}$ is exactly the homomorphism $\rho_A^{S,S'}$ defined in (1.4.1).

(1.5.2). Under the hypotheses of (1.5.1), let M be an A -module. There exists a canonical functorial morphism

$$\sigma : S'^{-1}(M_{[\phi]}) \longrightarrow (S^{-1}M)_{[\phi^{S}]}$$

of $S'^{-1}A'$ -modules, sending each element m/s' of $S'^{-1}(M_{[\phi]})$ to the element $m/\phi(s')$ of $(S^{-1}M)_{[\phi^{S}]}$; indeed, we immediately see that this definition does not depend on the representative m/s' of the element in question. When $S = \phi(S')$, the homomorphism σ is *bijective*. When $A' = A$ and ϕ is the identity, σ is none other than the homomorphism $\rho_M^{S,S'}$ defined in (1.4.1).

When, in particular, we take $M = A$ the homomorphism ϕ defines an A' -algebra structure on A ; $S'^{-1}(A_{[\phi]})$ is then endowed with a ring structure, with which it can be identified with $(\phi(S'))^{-1}A$, and the homomorphism $\sigma : S'^{-1}(A_{[\phi]}) \rightarrow S^{-1}A$ is a homomorphism of $S'^{-1}A'$ -algebras.

(1.5.3). Let M and N be A -modules; by composing the homomorphisms defined in (1.3.4) and (1.5.2), we obtain a homomorphism

$$(S^{-1}M \otimes_{S^{-1}A} S^{-1}N)_{[\phi^{S}]} \longleftarrow S'^{-1}((M \otimes N)_{[\phi]})$$

which is an isomorphism when $\phi(S') = S$. Similarly, by composing the homomorphisms in (1.3.5) and (1.5.2), we obtain a homomorphism

$$S'^{-1}((\text{Hom}_A(M, N))_{[\phi]}) \longrightarrow (\text{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N))_{[\phi^{S}]}$$

which is an isomorphism when $\phi(S') = S$ and M admits a finite presentation.

(1.5.4). Let us now consider an A' -module N' , and form the tensor product $N' \otimes_{A'} A_{[\phi]}$, which can be considered as an A -module by setting $a \cdot (n' \otimes b) = n' \otimes (ab)$. There is a functorial isomorphism of $S^{-1}A$ -modules

$$\tau : (S'^{-1}N') \otimes_{S'^{-1}A'} (S^{-1}A)_{[\phi^{S'}]} \simeq S^{-1}(N' \otimes_{A'} A_{[\phi]})$$

which sends the element $(n'/s') \otimes (a/s)$ to the element $(n' \otimes a)/(\phi(s')s)$; indeed, we can show that when we replace n'/s' (resp. a/s) by another expression of the same element, $(n' \otimes a)/(\phi(s')s)$ does not change; on the other hand, we can define a homomorphism inverse to τ by sending $(n' \otimes a)/s$ to the element $(n'/1) \otimes (a/s)$: we use the fact that $S^{-1}(N' \otimes_{A'} A_{[\phi]})$ is canonically isomorphic to $(N' \otimes_{A'} A_{[\phi]}) \otimes_A S^{-1}A$ (1.2.5), so also to $N' \otimes_{A'} (S^{-1}A)_{[\psi]}$, where we denote by ψ the composite homomorphism $a' \mapsto \phi(a')/1$ from A' to $S^{-1}A$.

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(1.5.5). If M' and N' are A' -modules, then by composing the isomorphisms (1.3.4) and (1.5.4), we obtain an isomorphism

$$S'^{-1}M' \otimes_{S'^{-1}A'} S'^{-1}N' \otimes_{S'^{-1}A'} S^{-1}A \simeq S^{-1}(M' \otimes_{A'} N' \otimes_{A'} A).$$

Likewise, if M' admits a finite presentation, we have by (1.3.5) and (1.5.4) an isomorphism

$$\text{Hom}_{S'^{-1}A'}(S'^{-1}M', S'^{-1}N') \otimes_{S'^{-1}A'} S^{-1}A \simeq S^{-1}(\text{Hom}_{A'}(M', N') \otimes_{A'} A).$$

(1.5.6). Under the hypotheses of (1.5.1), let T (resp. T') be another multiplicative subset of A (resp. A') such that $S \subset T$ (resp. $S' \subset T'$) and $\phi(T') \subset T$. Then the diagram

$$\begin{array}{ccc} S'^{-1}A' & \xrightarrow{\phi^{S'}} & S^{-1}A \\ \rho^{T',S'} \downarrow & & \downarrow \rho^{T,S} \\ T'^{-1}A' & \xrightarrow{\phi^{T'}} & T^{-1}A \end{array}$$

is commutative. If M is an A -module, then the diagram

$$\begin{array}{ccc} S'^{-1}(M_{[\phi]}) & \xrightarrow{\sigma} & (S^{-1}M)_{[\phi^{S'}]} \\ \rho^{T',S'} \downarrow & & \downarrow \rho^{T,S} \\ T'^{-1}(M_{[\phi]}) & \xrightarrow{\sigma} & (T^{-1}M)_{[\phi^{T'}]} \end{array}$$

is commutative. Finally, if N' is an A' -module, then the diagram

$$\begin{array}{ccc} (S'^{-1}N') \otimes_{S'^{-1}A'} (S^{-1}A)_{[\phi^{S'}]} & \xrightarrow{\sim} & S^{-1}(N' \otimes_{A'} A_{[\phi]}) \\ \downarrow & & \downarrow \rho^{T,S} \\ (T'^{-1}N') \otimes_{T'^{-1}A'} (T^{-1}A)_{[\phi^{T'}]} & \xrightarrow{\sim} & T^{-1}(N' \otimes_{A'} A_{[\phi]}) \end{array}$$

is commutative, the left vertical arrow obtained by applying $\rho_{N'}^{T',S'}$ to $S'^{-1}N'$ and $\rho_A^{T,S}$ to $S^{-1}A$.

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(1.5.7). Let A'' be a third ring, $\phi' : A'' \rightarrow A'$ a ring homomorphism, and S'' a multiplicative subset of A'' such that $\phi'(S'') \subset S'$. Let $\phi'' = \phi \circ \phi'$; then we have

$$\phi''^{S''} = \phi^{S'} \circ \phi'^{S''}.$$

Let M be an A -module; evidently we have $M_{[\phi'']} = (M_{[\phi]})_{[\phi'']}$; if σ' and σ'' are the homomorphisms defined by ϕ' and ϕ'' in the same way as how σ is defined in (1.5.2) by ϕ , then we have the transitivity formula

$$\sigma'' = \sigma \circ \sigma'.$$

Finally, let N'' be an A'' -module; the A -module $N'' \otimes_{A''} A_{[\phi'']}$ is canonically identified with $(N'' \otimes_{A''} A'_{[\phi']}) \otimes_{A'} A_{[\phi]}$, and likewise the $S^{-1}A$ -module $(S''^{-1}N'') \otimes_{S''^{-1}A''} (S^{-1}A)_{[\phi''s'']}$ is canonically identified with $((S''^{-1}N'') \otimes_{S''^{-1}A''} (S'^{-1}A')_{[\phi's'']}) \otimes_{S'^{-1}A'} (S^{-1}A)_{[\phi's']}$. With these identifications, if τ' and τ'' are the isomorphisms defined by ϕ' and ϕ'' in the same way as how τ is defined in (1.5.4) by ϕ , then we have the transitivity formula

$$\tau'' = \tau \circ (\tau' \otimes 1).$$

(1.5.8). Let A be a subring of a ring B ; for every *minimal* prime ideal \mathfrak{p} of A , there exists a minimal prime ideal \mathfrak{q} of B such that $\mathfrak{p} = A \cap \mathfrak{q}$. Indeed, $A_{\mathfrak{p}}$ is a subring of $B_{\mathfrak{p}}$ (1.3.2) and has a *single prime ideal* \mathfrak{p}' (1.2.6); since $B_{\mathfrak{p}}$ is not 0, it has at least one prime ideal \mathfrak{q}' and we necessarily have $\mathfrak{q}' \cap A_{\mathfrak{p}} = \mathfrak{p}'$; the prime ideal \mathfrak{q}_1 of B , the inverse image of \mathfrak{q}' , is thus such that $\mathfrak{q}_1 \cap A = \mathfrak{p}$, and *a fortiori* we have $\mathfrak{q} \cap A = \mathfrak{p}$ for every minimal prime ideal \mathfrak{q} of B contained in \mathfrak{q}_1 .

1.6. Identification of the module M_f as an inductive limit

(1.6.1). Let M be an A -module and f an element of A . Consider a sequence (M_n) of A -modules, all identical to M , and for each pair of integers $m \leq n$, let ϕ_{nm} be the homomorphism $z \mapsto f^{n-m}z$ from M_m to M_n ; it is immediate that $((M_n), (\phi_{nm}))$ is an *inductive system* of A -modules; let $N = \varinjlim M_n$ be the inductive limit of this system. We define a canonical *functorial* A -isomorphism from N to M_f . For this, let us note that, for all n , $\theta_n : z \mapsto z/f^n$ is an A -homomorphism from $M = M_n$ to M_f , and it follows from the definitions that we have $\theta_n \circ \phi_{nm} = \theta_m$ for $m \leq n$. As a result, there exists an A -homomorphism $\theta : N \rightarrow M_f$ such that, if ϕ_n denotes the canonical homomorphism $M_n \rightarrow N$, then we have $\theta_n = \theta \circ \phi_n$ for all n . Since, by hypothesis, every element of M_f is of the form z/f^n for at least one n , it is clear that θ is surjective. On the other hand, if $\theta(\phi_n(z)) = 0$, or, in other words, if $z/f^n = 0$, then there exists an integer $k > 0$ such that $f^k z = 0$, so $\phi_{n+k,n}(z) = 0$, which gives $\phi_n(z) = 0$. We can therefore identify M_f with $\varinjlim M_n$ via θ .

(1.6.2). Now write $M_{f,n}$, ϕ_{nm}^f , and ϕ_n^f instead of M_n , ϕ_{nm} , and ϕ_n . Let g be another element of A . Since f^n divides $f^n g^n$, we have a functorial homomorphism

$$\rho_{fg,f} : M_f \longrightarrow M_{fg} \quad ((1.4.1) \text{ and } (1.4.3));$$

if we identify M_f and M_{fg} with $\varinjlim M_{f,n}$ and $\varinjlim M_{fg,n}$ respectively, then $\rho_{fg,f}$ identifies with the *inductive limit* of the maps $\rho_{fg,f}^n : M_{f,n} \rightarrow M_{fg,n}$, defined by $\rho_{fg,f}^n(z) = g^n z$. Indeed, this follows immediately from the commutativity of the diagram

$$\begin{array}{ccc} M_{f,n} & \xrightarrow{\rho_{fg,f}^n} & M_{fg,n} \\ \phi_n^f \downarrow & & \downarrow \phi_n^{fg} \\ M_f & \xrightarrow{\rho_{fg,f}} & M_{fg}. \end{array}$$

1.7. Support of a module

(1.7.1). Given an A -module M , we define the *support* of M , denoted by $\text{Supp}(M)$, to be the set of prime ideals \mathfrak{p} of A such that $M_{\mathfrak{p}} \neq 0$. For it to be the case that $M = 0$, it is necessary and sufficient that $\text{Supp}(M) = \emptyset$, because if $M_{\mathfrak{p}} = 0$ for all \mathfrak{p} , then the annihilator of an element $x \in M$ cannot be contained in any prime ideal of A , and so is the whole of A .

(1.7.2). If $0 \rightarrow N \rightarrow M \rightarrow P \rightarrow 0$ is an exact sequence of A -modules, then we have

$$\text{Supp}(M) = \text{Supp}(N) \cup \text{Supp}(P)$$

because, for every prime ideal \mathfrak{p} of A , the sequence $0 \rightarrow N_{\mathfrak{p}} \rightarrow M_{\mathfrak{p}} \rightarrow P_{\mathfrak{p}} \rightarrow 0$ is exact (1.3.2) and in order that $M_{\mathfrak{p}} = 0$, it is necessary and sufficient that $N_{\mathfrak{p}} = P_{\mathfrak{p}} = 0$.

(1.7.3). If M is the sum of a family (M_{λ}) of submodules, then $M_{\mathfrak{p}}$ is the sum of the $(M_{\lambda})_{\mathfrak{p}}$ for every prime ideal \mathfrak{p} of A ((1.3.3) and (1.3.2)), so $\text{Supp}(M) = \bigcup_{\lambda} \text{Supp}(M_{\lambda})$.

(1.7.4). If M is an A -module of finite type, then $\text{Supp}(M)$ is the set of prime ideals containing the annihilator of M . Indeed, if M is cyclic and generated by x , then to say that $M_{\mathfrak{p}} = 0$ is to say that there exists an $s \notin \mathfrak{p}$ such that $s \cdot x = 0$, and thus that \mathfrak{p} does not contain the annihilator of x . Now if M admits a finite system $(x_i)_{1 \leq i \leq n}$ of generators, and if \mathfrak{a}_i is the annihilator of x_i , then it follows from (1.7.3) that $\text{Supp}(M)$ is the set of the \mathfrak{p} containing one of the \mathfrak{a}_i , or equivalently, the set of the \mathfrak{p} containing $\mathfrak{a} = \bigcap_i \mathfrak{a}_i$, which is the annihilator of M .

(1.7.5). If M and N are two A -modules of finite type, then we have

$$\text{Supp}(M \otimes_A N) = \text{Supp}(M) \cap \text{Supp}(N).$$

It is a question of seeing that, if \mathfrak{p} is a prime ideal of A , then the condition $M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} N_{\mathfrak{p}} \neq 0$ is equivalent to “ $M_{\mathfrak{p}} \neq 0$ and $N_{\mathfrak{p}} \neq 0$ ” (taking (1.3.4) into account). In other words, it is a question of seeing that, if P and Q are modules of finite type over a local ring $B \neq 0$, then $P \otimes_B Q \neq 0$. Let \mathfrak{m} be the maximal ideal of B . By Nakayama’s Lemma, the vector spaces $P/\mathfrak{m}P$ and $Q/\mathfrak{m}Q$ are not 0, and so it is the same for the tensor product $(P/\mathfrak{m}P) \otimes_{B/\mathfrak{m}} (Q/\mathfrak{m}Q) = (P \otimes_B Q) \otimes_B (B/\mathfrak{m})$, whence the conclusion.

In particular, if M is an A -module of finite type, and \mathfrak{a} an ideal of A , then $\text{Supp}(M/\mathfrak{a}M)$ is the set of prime ideals containing both \mathfrak{a} and the annihilator \mathfrak{n} of M (1.7.4), that is, the set of prime ideals containing $\mathfrak{a} + \mathfrak{n}$.

§2. IRREDUCIBLE SPACES. NOETHERIAN SPACES

2.1. Irreducible spaces

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(2.1.1). We say that a topological space X is *irreducible* if it is nonempty and if it is not a union of two distinct closed subspaces of X . It is equivalent to say that $X \neq \emptyset$ and the intersection of two nonempty open sets (and consequently of a finite number of open sets) of X is nonempty, or that every nonempty open set is everywhere dense, or that any closed set is *rare*¹, or, lastly, that all open sets of X are *connected*.

(2.1.2). For a subspace Y of a topological space X to be irreducible, it is necessary and sufficient that its closure \overline{Y} be irreducible. In particular, any subspace which is the closure $\overline{\{x\}}$ of a singleton is irreducible; we will express the relation $y \in \overline{\{x\}}$ (equivalent to $\overline{\{y\}} \subset \overline{\{x\}}$) by saying that y is a *specialization* of x or that x is a *generalization* of y . When there exists, in an irreducible space X , a point x such that $X = \overline{\{x\}}$, we will say that x is a *generic point* of X . Any nonempty open subset of X then contains x , and any subspace containing x has x as a generic point.

(2.1.3). Recall that a *Kolmogoroff space* is a topological space X satisfying the axiom of separation:

(T_0) If $x \neq y$ are any two points of X , there is an open set containing one of the points x and y , but not the other.

If an irreducible Kolmogoroff space admits a generic point, it admits *exactly* one, since a nonempty open set contains any generic point.

Recall that a topological space X is said to be *quasi-compact* if, from any collection of open sets of X , one can extract a finite cover of X (or, equivalently, if any decreasing filtered family of nonempty closed sets has a nonempty intersection). If X is a quasi-compact space, then any nonempty closed subset A of X contains a *minimal* nonempty closed set M , because the set of nonempty closed subsets of A is inductive under the relation \supset ; if, in addition, X is a Kolmogoroff space, M is necessarily a single point (or, as we say by abuse of language, is a *closed point*).

(2.1.4). In an irreducible space X , every nonempty open subspace U is irreducible, and if X admits a generic point x , x is also a generic point of U .

To prove this, let (U_{α}) be a cover (whose set of indices is nonempty) of a topological space X , consisting of nonempty open sets; if X is irreducible, it is necessary and sufficient that U_{α} is irreducible for all α , and that $U_{\alpha} \cap U_{\beta} \neq \emptyset$ for any α, β . The condition is clearly necessary; to see that it is sufficient, it suffices to prove that if V is a nonempty open subset of X , then $V \cap U_{\alpha}$ is nonempty for all α , since then $V \cap U_{\alpha}$ is dense in U_{α} for all α , and consequently V is dense in X . Now there is

¹[Trans] also known as *nowhere dense*.

at least one index γ such that $V \cap U_\gamma \neq \emptyset$, so $V \cap U_\gamma$ is dense in U_γ , and as for all α , $U_\alpha \cap V_\alpha \neq \emptyset$, we also have $V \cap U_\alpha \cap U_\gamma \neq \emptyset$.

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(2.1.5). Let X be an irreducible space, and f a continuous map from X into a topological space Y . Then $f(X)$ is irreducible, and if x is a generic point of X , then $f(x)$ is a generic point of $f(X)$ and hence also of $\overline{f(X)}$. In particular, if, in addition, Y is irreducible and with a single generic point y , then for $f(X)$ to be everywhere dense, it is necessary and sufficient that $f(x) = y$.

(2.1.6). Any irreducible subspace of a topological space X is contained in a maximal irreducible subspace, which is necessarily closed. Maximal irreducible subspaces of X are called the *irreducible components* of X . If Z_1 and Z_2 are two irreducible components distinct from the space X , then $Z_1 \cap Z_2$ is a closed *rare* set in each of the subspaces Z_1, Z_2 ; in particular, if an irreducible component of X admits a generic point (2.1.2), such a point cannot belong to any other irreducible component. If X has only a *finite* number of irreducible components Z_i ($1 \leq i \leq n$), and if, for each i , we put $U_i = \bigcup_{j \neq i} Z_j$, then the U_i are open, irreducible, disjoint, and their union is dense in X . Let U be an open subset of a topological space X . If Z is an irreducible subset of X that intersects U , then $Z \cap U$ is open and dense in Z , thus irreducible; conversely, for any irreducible closed subset Y of U , the closure \bar{Y} of Y in X is irreducible and $\bar{Y} \cap U = Y$. We conclude that there is a *bijective correspondence* between the irreducible components of U and the irreducible components of X which intersect U .

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(2.1.7). If a topological space X is a union of a *finite* number of irreducible closed subspaces Y_i , then the irreducible components of X are the maximal elements of the set of the Y_i , because if Z is an irreducible closed subset of X , then Z is the union of the $Z \cap Y_i$, from which one sees that Z must be contained in one of the Y_i . Let Y be a subspace of a topological space X , and suppose that Y has only a finite number of irreducible components Y_i , ($1 \leq i \leq n$); then the closures \bar{Y}_i in X are the irreducible components of Y .

(2.1.8). Let Y be an irreducible space admitting a single generic point y . Let X be a topological space, and f a continuous map from X to Y . Then, for any irreducible component Z of X intersecting $f^{-1}(y)$, $f(Z)$ is dense in Y . The converse is not necessarily true; however, if Z has a generic point z , and if $f(Z)$ is dense in Y , then we must have $f(z) = y$ (2.1.5); in addition, $Z \cap f^{-1}(y)$ is then the closure of $\{z\}$ in $f^{-1}(y)$ and is therefore irreducible, and as an irreducible subset of $f^{-1}(y)$ containing z is necessarily contained in Z (2.1.6), z is a generic point of $Z \cap f^{-1}(y)$. As any irreducible component of $f^{-1}(y)$ is contained in an irreducible component of X , we see that, if any irreducible component Z of X intersecting $f^{-1}(y)$ admits a generic point, then there is a *bijective correspondence* between all these components and all the irreducible components $Z \cap f^{-1}(y)$ of $f^{-1}(y)$, the generic points of Z being identical to those of $Z \cap f^{-1}(y)$.

2.2. Noetherian spaces

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(2.2.1). We say that a topological space X is *Noetherian* if the set of open subsets of X satisfies the *maximal* condition, or, equivalently, if the set of closed subsets of X satisfies the *minimal* condition. We say that X is *locally Noetherian* if each $x \in X$ admits a neighbourhood which is a Noetherian subspace.

(2.2.2). Let E be an ordered set satisfying the *minimal* condition, and let \mathbf{P} be a property of the elements of E subject to the following condition: if $a \in E$ is such that for any $x < a$, $\mathbf{P}(x)$ is true, then $\mathbf{P}(a)$ is true. Under these conditions, $\mathbf{P}(x)$ is true for all $x \in E$ ("principle of Noetherian induction"). Indeed, let F be the set of $x \in E$ for which $\mathbf{P}(x)$ is false; if F were not empty, it would have a minimal element a , and as then $\mathbf{P}(x)$ is true for all $x < a$, $\mathbf{P}(a)$ would be true, which is a contradiction.

We will apply this principle in particular when E is a *set of closed subsets of a Noetherian space*.

(2.2.3). Any subspace of a Noetherian space is Noetherian. Conversely, any topological space that is a finite union of Noetherian subspaces is Noetherian.

(2.2.4). Any Noetherian space is quasi-compact; conversely, any topological space in which all open sets are quasi-compact is Noetherian.

(2.2.5). A Noetherian space has only a *finite* number of irreducible components, as we see by Noetherian induction.

§3. SUPPLEMENT ON SHEAVES

3.1. Sheaves with values in a category

(3.1.1). Let \mathcal{C} be a category, $(A_\alpha)_{\alpha \in I}$, $(A_{\alpha\beta})_{(\alpha,\beta) \in I \times I}$ two families of objects of \mathcal{C} such that $A_{\beta\alpha} = A_{\alpha\beta}$, and $(\rho_{\alpha\beta})_{(\alpha,\beta) \in I \times I}$ a family of morphisms $\rho_{\alpha\beta} : A_\alpha \rightarrow A_{\alpha\beta}$. We say that a pair consisting of an object A of \mathcal{C} and a family of morphisms $\rho_\alpha : A \rightarrow A_\alpha$ is a *solution to the universal problem* defined by the data of the families (A_α) , $(A_{\alpha\beta})$, and $(\rho_{\alpha\beta})$ if, for every object B of \mathcal{C} , the map which sends $f \in \text{Hom}(B, A)$ to the family $(\rho_\alpha \circ f) \in \prod_\alpha \text{Hom}(B, A_\alpha)$ is a *bijection* of $\text{Hom}(B, A)$ to the set of all (f_α) such that $\rho_{\alpha\beta} \circ f_\alpha = \rho_{\beta\alpha} \circ f_\beta$ for any pair of indices (α, β) . If such a solution exists, it is unique up to an isomorphism.

(3.1.2). We will not recall the definition of a *presheaf* $U \mapsto \mathcal{F}(U)$ on a topological space X with values in a category \mathcal{C} (G, I, 1.9); we say that such a presheaf is a *sheaf with values in \mathcal{C}* if it satisfies the following axiom:

(F) For any covering (U_α) of an open set U of X by open sets U_α contained in U , if we denote by ρ_α (resp. $\rho_{\alpha\beta}$) the restriction morphism

$$\mathcal{F}(U) \longrightarrow \mathcal{F}(U_\alpha) \quad (\text{resp. } \mathcal{F}(U_\alpha) \longrightarrow \mathcal{F}(U_\alpha \cap U_\beta)),$$

the pair formed by $\mathcal{F}(U)$ and the family (ρ_α) are a solution to the universal problem for $(\mathcal{F}(U_\alpha))$, $(\mathcal{F}(U_\alpha \cap U_\beta))$, and $(\rho_{\alpha\beta})$ in (3.1.1)².

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Equivalently, we can say that, for each object T of \mathcal{C} , that the family $U \mapsto \text{Hom}(T, \mathcal{F}(U))$ is a *sheaf of sets*.

(3.1.3). Assume that \mathcal{C} is the category defined by a “type of structure with morphisms” Σ , the objects of \mathcal{C} being the sets with structures of type Σ and morphisms those of Σ . Suppose that the category \mathcal{C} also satisfies the following condition:

(E) If $(A, (\rho_\alpha))$ is a solution of a universal mapping problem in the category \mathcal{C} for families (A_α) , $(A_{\alpha\beta})$, $(\rho_{\alpha\beta})$, then it is also a solution of the universal mapping problem for the same families in the category of sets (that is, when we consider A , A_α , and $A_{\alpha\beta}$ as sets, ρ_α and $\rho_{\alpha\beta}$ as functions)³.

Under these conditions, the condition (F) gives that, when considered as a presheaf of sets, $U \mapsto \mathcal{F}(U)$ is a *sheaf*. In addition, for a map $u : T \rightarrow \mathcal{F}(U)$ to be a morphism of \mathcal{C} , it is necessary and sufficient, according to (F), that each map $\rho_\alpha \circ u$ is a morphism $T \rightarrow \mathcal{F}(U_\alpha)$, which means that the structure of type Σ on $\mathcal{F}(U)$ is the *initial structure* for the morphisms ρ_α . Conversely, suppose a presheaf $U \mapsto \mathcal{F}(U)$ on X , with values in \mathcal{C} , is a *sheaf of sets* and satisfies the previous condition; it is then clear that it satisfies (F), so it is a *sheaf with values in \mathcal{C}* .

(3.1.4). When Σ is a type of a group or ring structure, the fact that the presheaf $U \mapsto \mathcal{F}(U)$ with values in \mathcal{C} is a sheaf of sets implies *ipso facto* that it is a sheaf with values in \mathcal{C} (in other words, a sheaf of groups or rings within the meaning of (G))⁴. But it is not the same when, for example, \mathcal{C} is the category of *topological rings* (with morphisms as continuous homomorphisms): a sheaf with values in \mathcal{C} is a sheaf of rings $U \mapsto \mathcal{F}(U)$ such that for any open U and any covering of U by open sets $U_\alpha \subset U$, the topology of the ring $\mathcal{F}(U)$ is to be the *least fine* making the homomorphisms $\mathcal{F}(U) \rightarrow \mathcal{F}(U_\alpha)$ continuous. We will say in this case that $U \mapsto \mathcal{F}(U)$, considered as a sheaf of rings (without a topology), is *underlying* the sheaf of topological rings $U \mapsto \mathcal{F}(U)$. Morphisms $u_V : \mathcal{F}(V) \rightarrow \mathcal{G}(V)$ (V an arbitrary open subset of X) of sheaves of topological rings are therefore homomorphisms of the underlying sheaves of rings, such that u_V is *continuous* for all open $V \subset X$; to distinguish them from any homomorphisms of the sheaves of the underlying rings, we will call

²This is a special case of the more general notion of a (non-filtered) *projective limit* (see (T, I, 1.8) and the book in preparation announced in the introduction).

³It can be proved that it also means that the canonical functor $\mathcal{C} \rightarrow \text{Set}$ commutes with projective limits (not necessarily filtered).

⁴This is because in the category \mathcal{C} , any morphism that is a *bijection* (as a map of sets) is an *isomorphism*. This is no longer true when \mathcal{C} is the category of topological spaces, for example.

them continuous homomorphisms of sheaves of topological rings. We have similar definitions and conventions for sheaves of topological spaces or topological groups.

(3.1.5). It is clear that for any category \mathcal{C} , if there is a presheaf (respectively a sheaf) \mathcal{F} on X with values in \mathcal{C} and U is an open set of X , the $\mathcal{F}(V)$ for open $V \subset U$ constitute a presheaf (or a sheaf) with values in \mathcal{C} , which we call the presheaf (or sheaf) *induced* by \mathcal{F} on U and denote it by $\mathcal{F}|U$.

For any morphism $u : \mathcal{F} \rightarrow \mathcal{G}$ of presheaves on X with values in \mathcal{C} , we denote by $u|U$ the morphism $\mathcal{F}|U \rightarrow \mathcal{G}|U$ consisting of the u_V for $V \subset U$.

(3.1.6). Suppose now that the category \mathcal{C} admits *inductive limits* (T, 1.8); then, for any presheaf (and in particular any sheaf) \mathcal{F} on X with values in \mathcal{C} and each $x \in X$, we can define the *stalk* \mathcal{F}_x as the object of \mathcal{C} defined by the inductive limit of the $\mathcal{F}(U)$ with respect to the filtered set (for \supset) of the open neighbourhoods U of x in X , and the morphisms $\rho_U^V : \mathcal{F}(V) \rightarrow \mathcal{F}(U)$. If $u : \mathcal{F} \rightarrow \mathcal{G}$ is a morphism of presheaves with values in \mathcal{C} , we define for each $x \in X$ the morphism $u_x : \mathcal{F}_x \rightarrow \mathcal{G}_x$ as the inductive limit of $u_U : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ with respect to all open neighbourhoods of x ; we thus define \mathcal{F}_x as a covariant functor in \mathcal{F} , with values in \mathcal{C} , for all $x \in X$.

When \mathcal{C} is further defined by a kind of structure with morphisms Σ , we call *sections over U* of a sheaf \mathcal{F} with values in \mathcal{C} the elements of $\mathcal{F}(U)$, and we write $\Gamma(U, \mathcal{F})$ instead of $\mathcal{F}(U)$; for $s \in \Gamma(U, \mathcal{F})$, V an open set contained in U , we write $s|V$ instead of $\rho_V^U(s)$; for all $x \in U$, the canonical image of s in \mathcal{F}_x is the *germ* of s at the point x , denoted by s_x (we will never replace the notation $s(x)$ in this sense, this notation being reserved for another notion relating to sheaves which will be considered in this treatise (5.5.1)).

If then $u : \mathcal{F} \rightarrow \mathcal{G}$ is a morphism of sheaves with values in \mathcal{C} , we will write $u(s)$ instead of $u_V(s)$ for all $s \in \Gamma(V, \mathcal{F})$.

If \mathcal{F} is a sheaf of commutative groups, or rings, or modules, we say that the set of $x \in X$ such that $\mathcal{F}_x \neq \{0\}$ is the *support* of \mathcal{F} , denoted $\text{Supp}(\mathcal{F})$; this set is not necessarily closed in X .

When \mathcal{C} is defined by a type of structure with morphisms, we *systematically refrain from using the point of view of "étalé spaces"* in terms of relating to sheaves with values in \mathcal{C} ; in other words, we will never consider a sheaf as a topological space (nor even as the whole union of its stalks), and we will not consider also a morphism $u : \mathcal{F} \rightarrow \mathcal{G}$ of such sheaves on X as a continuous map of topological spaces.

3.2. Presheaves on an open basis

(3.2.1). We will restrict to the following categories \mathcal{C} admitting *projective limits* (generalized, that is, corresponding to not necessarily filtered preordered sets, cf. (T, 1.8)). Let X be a topological space, \mathfrak{B} an open basis for the topology of X . We will call a *presheaf on \mathfrak{B} , with values in \mathcal{C}* , a family of objects $\mathcal{F}(U) \in \mathcal{C}$, corresponding to each $U \in \mathfrak{B}$, and a family of morphisms $\rho_U^V : \mathcal{F}(V) \rightarrow \mathcal{F}(U)$ defined for any pair (U, V) of elements of \mathfrak{B} such that $U \subset V$, with the conditions $\rho_U^U = \text{identity}$ and $\rho_U^W = \rho_U^V \circ \rho_V^W$ if U, V, W in \mathfrak{B} are such that $U \subset V \subset W$. We can associate a *presheaf with values in \mathcal{C}* : $U \mapsto \mathcal{F}(U)$ in the ordinary sense, taking for all open U , $\mathcal{F}'(U) = \varprojlim \mathcal{F}(V)$, where V runs through the ordered set (for \subset , *not filtered* in general) of $V \in \mathfrak{B}$ sets such that $V \subset U$, since the (V) form a projective system for the ρ_V^W ($V \subset W \subset U$, $V \in \mathfrak{B}$, $W \in \mathfrak{B}$). Indeed, if U, U' are two open sets of X such that $U \subset U'$, we define $\rho_U^{U'}$ as the projective limit (for $V \subset U$) of the canonical morphisms $\mathcal{F}'(U') \rightarrow \mathcal{F}(V)$, in other words the unique morphism $\mathcal{F}'(U') \rightarrow \mathcal{F}'(U)$, which, when composed with the canonical morphisms $\mathcal{F}'(U) \rightarrow \mathcal{F}(V)$, gives the canonical morphisms $\mathcal{F}'(U') \rightarrow \mathcal{F}(V)$; the verification of the transitivity of $\rho_U^{U'}$ is then immediate. Moreover, if $U \in \mathfrak{B}$, the canonical morphism $\mathcal{F}'(U) \rightarrow \mathcal{F}(U)$ is an isomorphism, allowing us to identify these two objects⁵.

⁵If X is a *Noetherian* space, we can still define $\mathcal{F}'(U)$ and show that it is a presheaf (in the ordinary sense) when one supposes only that \mathcal{C} admits projective limits for *finite* projective systems. Indeed, if U is any open set of X , there is a *finite* covering (V_i) of U consisting of sets of \mathfrak{B} ; for every couple (i, j) of indices, let (V_{ijk}) be a finite covering of $V_i \cap V_j$ formed by sets of \mathfrak{B} . Let I be the set of i and triples (i, j, k) , ordered only by the relations $i > (i, j, k)$, $j > (i, j, k)$; we then take $\mathcal{F}'(U)$ to be the projective limit of the system of $\mathcal{F}(V_i)$ and $\mathcal{F}(V_{ijk})$; it is easy to verify that this does not depend on the coverings (V_i) and (V_{ijk}) and that $U \mapsto \mathcal{F}'(U)$ is a presheaf.

(3.2.2). For the presheaf \mathcal{F}' thus defined to be a *sheaf*, it is necessary and sufficient that the presheaf \mathcal{F} on \mathfrak{B} satisfies the condition:

- (F₀) For any covering (U_α) of $U \in \mathfrak{B}$ by sets $U_\alpha \in \mathfrak{B}$ contained in U , and for any object $T \in \mathcal{C}$, the map which sends $f \in \text{Hom}(T, \mathcal{F}(U))$ to the family $(\rho_{U_\alpha}^U \circ f) \in \prod_\alpha \text{Hom}(T, \mathcal{F}(U_\alpha))$ is a bijection from $\text{Hom}(T, \mathcal{F}(U))$ to the set of all (f_α) such that $\rho_V^{U_\alpha} \circ f_\alpha = \rho_V^{U_\beta} \circ f_\beta$ for any pair of indices (α, β) and any $V \in \mathfrak{B}$ such that $V \subset U_\alpha \cap U_\beta$ ⁶.

The condition is obviously necessary. To show that it is sufficient, consider first a second basis \mathfrak{B}' of the topology of X , contained in \mathfrak{B} , and show that if \mathcal{F}'' denotes the presheaf induced by the subfamily $(\mathcal{F}(V))_{V \in \mathfrak{B}'}$, \mathcal{F}'' is canonically isomorphic to \mathcal{F}' . Indeed, first the projective limit (for $V \in \mathfrak{B}'$, $V \subset U$) of the canonical morphisms $\mathcal{F}'(U) \rightarrow \mathcal{F}(V)$ is a morphism $\mathcal{F}'(U) \rightarrow \mathcal{F}''(U)$ for all open U . If $U \in \mathfrak{B}$, this morphism is an isomorphism, because by hypothesis the canonical morphisms $\mathcal{F}''(U) \rightarrow \mathcal{F}(V)$ for $V \in \mathfrak{B}'$, $V \subset U$, factorize as $\mathcal{F}''(U) \rightarrow \mathcal{F}(U) \rightarrow \mathcal{F}(V)$, and it is immediate to see that the composition of morphisms $\mathcal{F}(U) \rightarrow \mathcal{F}''(U)$ and $\mathcal{F}''(U) \rightarrow \mathcal{F}(U)$ thus defined are the identities. This being so, for all open U , the morphisms $\mathcal{F}''(U) \rightarrow \mathcal{F}''(W) = \mathcal{F}(W)$ for $W \in \mathfrak{B}$ and $W \subset U$ satisfy the conditions characterizing the projective limit of $\mathcal{F}(W)$ ($W \in \mathfrak{B}$, $W \subset U$), which proves our assertion given the uniqueness of a projective limit up to isomorphism.

This being so, let U be any open set of X , (U_α) a covering of U by the open sets contained in U , and \mathfrak{B}' the subfamily of \mathfrak{B} formed by the sets of \mathfrak{B} contained in at least one U_α ; it is clear that \mathfrak{B}' is still a basis of the topology of U , so $\mathcal{F}'(U)$ (resp. $\mathcal{F}''(U_\alpha)$) is the projective limit of $\mathcal{F}(V)$ for $V \in \mathfrak{B}'$ and $V \subset U$ (resp., $V \subset U_\alpha$), the axiom (F) is then immediately verified by virtue of the definition of the projective limit.

When (F₀) is satisfied, we will say by abuse of language that the presheaf \mathcal{F} on the basis \mathfrak{B} is a sheaf.

(3.2.3). Let \mathcal{F}, \mathcal{G} be two presheaves on a basis \mathfrak{B} , with values in \mathcal{C} ; we define a *morphism* $u : \mathcal{F} \rightarrow \mathcal{G}$ as a family $(u_V)_{V \in \mathfrak{B}}$ of morphisms $u_V : \mathcal{F}(V) \rightarrow \mathcal{G}(V)$ satisfying the usual compatibility conditions with the restriction morphisms ρ_V^W . With the notation of (3.2.1), we have a morphism $u' : \mathcal{F}' \rightarrow \mathcal{G}'$ of (ordinary) presheaves by taking for u'_U the projective limit of the u_V for $V \in \mathfrak{B}$ and $V \subset U$; the verification of the compatibility conditions with the ρ'^U_U follows from the functorial properties of the projective limit.

(3.2.4). If the category \mathcal{C} admits inductive limits, and if \mathcal{F} is a presheaf on the basis \mathfrak{B} , with values in \mathcal{C} , for each $x \in X$ the neighbourhoods of x belonging to \mathfrak{B} form a cofinal set (for \supset) in the set of neighbourhoods of x , therefore, if \mathcal{F}' is the (ordinary) presheaf corresponding to \mathcal{F} , the stalk \mathcal{F}'_x is equal to $\varinjlim_{\mathfrak{B}} \mathcal{F}(V)$ over the set of $V \in \mathfrak{B}$ containing x . If $u : \mathcal{F} \rightarrow \mathcal{G}$ is morphism of presheaves on \mathfrak{B} with values in \mathcal{C} , $u' : \mathcal{F}' \rightarrow \mathcal{G}'$ the corresponding morphism of ordinary presheaves, u'_x is likewise the inductive limit of the morphisms $u_V : \mathcal{F}(V) \rightarrow \mathcal{G}(V)$ for $V \in \mathfrak{B}$, $x \in V$.

(3.2.5). We return to the general conditions of (3.2.1). If \mathcal{F} is an ordinary *sheaf* with values in \mathcal{C} , \mathcal{F}_1 the sheaf on \mathfrak{B} obtained by the restriction of \mathcal{F} to \mathfrak{B} , then the ordinary sheaf \mathcal{F}'_1 obtained from \mathcal{F}_1 by the procedure of (3.2.1) is canonically isomorphic to \mathcal{F} , by virtue of the condition (F) and the uniqueness properties of the projective limit. We identify the ordinary sheaf \mathcal{F} with \mathcal{F}'_1 .

If \mathcal{G} is a second (ordinary) sheaf on X with values in \mathcal{C} , and $u : \mathcal{F} \rightarrow \mathcal{G}$ a morphism, the preceding remark shows that the data of the $u_V : \mathcal{F}(V) \rightarrow \mathcal{G}(V)$ for only the $V \in \mathfrak{B}$ completely determines u ; conversely, it is sufficient, the u_V being given for $V \in \mathfrak{B}$, to verify the commutative diagram with the restriction morphisms ρ_V^W for $V \in \mathfrak{B}$, $W \in \mathfrak{B}$, and $V \subset W$, for there to exist a morphism u' and a unique \mathcal{F} in \mathcal{G} such that $u'_V = u_V$ for each $V \in \mathfrak{B}$ (3.2.3).

(3.2.6). Suppose that \mathcal{C} admits projective limits. Then the category of *sheaves on X with values in \mathcal{C}* admits *projective limits*; if (\mathcal{F}_λ) is a projective system of sheaves on X with values in \mathcal{C} , the $\mathcal{F}(U) = \varprojlim_\lambda \mathcal{F}_\lambda(U)$ indeed define a presheaf with values in \mathcal{C} , and the verification of the axiom (F)

⁶It also means that the pair formed by $\mathcal{F}(U)$ and the $\rho_\alpha = \rho_{U_\alpha}^U$ is a *solution to the universal problem* defined in (3.1.1) by the data of $A_\alpha = \mathcal{F}(U_\alpha)$, $A_{\alpha\beta} = \Pi \mathcal{F}(V)$ (for $V \in \mathfrak{B}$ such that $V \subset U_\alpha \cap U_\beta$) and $\rho_{\alpha\beta} = (\rho_V^U) : \mathcal{F}(U_\alpha) \rightarrow \Pi \mathcal{F}(V)$ defined by the condition that for $V \in \mathfrak{B}$, $V' \in \mathfrak{B}$, $W \in \mathfrak{B}$, $V \cup V' \subset U_\alpha \cap U_\beta$, $W \subset V \cap V'$, $\rho_W^V \circ \rho_V^U = \rho_W^{V'} \circ \rho_{V'}^U$.

follows from the transitivity of projective limits; the fact that \mathcal{F} is then the projective limit of the \mathcal{F}_λ is immediate.

When \mathbf{C} is the category of sets, for each projective system (\mathcal{H}_λ) such that \mathcal{H}_λ is a *subsheaf* of \mathcal{F}_λ for each λ , $\varprojlim_\lambda \mathcal{H}_\lambda$ canonically identifies with a *subsheaf* of $\varprojlim_\lambda \mathcal{F}_\lambda$. If \mathbf{C} is the category of abelian groups, the covariant functor $\varprojlim_\lambda \mathcal{F}_\lambda$ is *additive* and *left exact*.

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3.3. Gluing sheaves

(3.3.1). Suppose still that the category \mathbf{C} admits (generalized) projective limits. Let X be a topological space, $\mathcal{U} = (U_\lambda)_{\lambda \in L}$ an open cover of X , and for each $\lambda \in L$, let \mathcal{F}_λ be a sheaf on U_λ , with values in \mathbf{C} ; for each pair of indices (λ, μ) , suppose that we are given an *isomorphism* $\theta_{\lambda\mu} : \mathcal{F}_\mu|_{(U_\lambda \cap U_\mu)} \simeq \mathcal{F}_\lambda|_{(U_\lambda \cap U_\mu)}$; in addition, suppose that for each triple (λ, μ, ν) , if we denote by $\theta'_{\lambda\mu}, \theta'_{\mu\nu}, \theta'_{\lambda\nu}$ the restrictions of $\theta_{\lambda\mu}, \theta_{\mu\nu}, \theta_{\lambda\nu}$ to $U_\lambda \cap U_\mu \cap U_\nu$, then we have $\theta'_{\lambda\nu} = \theta'_{\lambda\mu} \circ \theta'_{\mu\nu}$ (*gluing condition* for the $\theta_{\lambda\mu}$). Then there exists a sheaf \mathcal{F} on X , with values in \mathbf{C} , and for each λ an isomorphism $\eta_\lambda : \mathcal{F}|_{U_\lambda} \simeq \mathcal{F}_\lambda$ such that, for each pair (λ, μ) , if we denote by η'_λ and η'_μ the restrictions of η_λ and η_μ to $U_\lambda \cap U_\mu$, then we have $\theta_{\lambda\mu} = \eta'_\lambda \circ \eta'^{-1}_\mu$; in addition, \mathcal{F} and the η_λ are determined up to unique isomorphism by these conditions. The uniqueness indeed follows immediately from (3.2.5). To establish the existence of \mathcal{F} , denote by \mathfrak{B} the open basis consisting of the open sets contained in at least one U_λ , and for each $U \in \mathfrak{B}$, choose (by the Hilbert function τ) one of the $\mathcal{F}_\lambda(U)$ for one of the λ such that $U \subset U_\lambda$; if we denote this object by $\mathcal{F}(U)$, the ρ_U^V for $U \subset V, U \in \mathfrak{B}, V \in \mathfrak{B}$ are defined in an evident way (by means of the $\theta_{\lambda\mu}$), and the transitivity condition is a consequence of the gluing condition; in addition, the verification of (F₀) is immediate, so the presheaf on \mathfrak{B} thus clearly defines a sheaf, and we deduce by the general procedure (3.2.1) an (ordinary) sheaf still denoted \mathcal{F} and which answers the question. We say that \mathcal{F} is obtained by *gluing the \mathcal{F}_λ by means of the $\theta_{\lambda\mu}$* and we usually identify the \mathcal{F}_λ and $\mathcal{F}|_{U_\lambda}$ by means of the η_λ .

It is clear that each sheaf \mathcal{F} on X with values in \mathbf{C} can be considered as being obtained by the gluing of the sheaves $\mathcal{F}_\lambda = \mathcal{F}|_{U_\lambda}$ (where (U_λ) is an arbitrary open cover of X), by means of the isomorphisms $\theta_{\lambda\mu}$ reduced to the identity.

(3.3.2). With the same notation, let \mathcal{G}_λ be a second sheaf on U_λ (for each $\lambda \in L$) with values in \mathbf{C} , and for each pair (λ, μ) let us be given an isomorphism $\omega_{\lambda\mu} : \mathcal{G}_\mu|_{(U_\lambda \cap U_\mu)} \simeq \mathcal{G}_\lambda|_{(U_\lambda \cap U_\mu)}$, these isomorphisms satisfying the gluing condition. Finally, suppose that we are given for each λ a morphism $u_\lambda : \mathcal{F}_\lambda \rightarrow \mathcal{G}_\lambda$, and that the diagrams

$$(3.3.2.1) \quad \begin{array}{ccc} \mathcal{F}_\mu|_{(U_\lambda \cap U_\mu)} & \xrightarrow{u_\mu} & \mathcal{G}_\mu|_{(U_\lambda \cap U_\mu)} \\ \downarrow & & \downarrow \\ \mathcal{F}_\lambda|_{(U_\lambda \cap U_\mu)} & \xrightarrow{u_\lambda} & \mathcal{G}_\lambda|_{(U_\lambda \cap U_\mu)} \end{array}$$

are commutative. Then, if \mathcal{G} is obtained by gluing the \mathcal{G}_λ by means of the $\omega_{\lambda\mu}$, there exists a unique morphism $u : \mathcal{F} \rightarrow \mathcal{G}$ such that the diagrams

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$$\begin{array}{ccc} \mathcal{F}|_{U_\lambda} & \xrightarrow{u|_{U_\lambda}} & \mathcal{G}|_{U_\lambda} \\ \downarrow & & \downarrow \\ \mathcal{F}_\lambda & \xrightarrow{u_\lambda} & \mathcal{G}_\lambda \end{array}$$

are commutative; this follows immediately from (3.2.3). The correspondence between the family (u_λ) and u is in a functorial bijection with the subset of $\prod_\lambda \text{Hom}(\mathcal{F}_\lambda, \mathcal{G}_\lambda)$ satisfying the conditions (3.3.2.1) on $\text{Hom}(\mathcal{F}, \mathcal{G})$.

(3.3.3). With the notation of (3.3.1), let V be an open set of X ; it is immediate that the restrictions to $V \cap U_\lambda \cap U_\mu$ of the $\theta_{\lambda\mu}$ satisfy the gluing condition for the induced sheaves $\mathcal{F}_\lambda|_{(V \cap U_\lambda)}$ and that the sheaves on V obtained by gluing the latter identifies canonically with $\mathcal{F}|_V$.

3.4. Direct images of presheaves

(3.4.1). Let X, Y be two topological spaces, $\psi : X \rightarrow Y$ a continuous map. Let \mathcal{F} be a presheaf on X with values in a category \mathcal{C} ; for each open $U \subset Y$, let $\mathcal{G}(U) = \mathcal{F}(\psi^{-1}(U))$, and if U, V are two open subsets of Y such that $U \subset V$, let ρ_U^V be the morphism $\mathcal{F}(\psi^{-1}(V)) \rightarrow \mathcal{F}(\psi^{-1}(U))$; it is immediate that the $\mathcal{G}(U)$ and the ρ_U^V define a *presheaf* on Y with values in \mathcal{C} , that we call the *direct image of \mathcal{F} by ψ* and we denote it by $\psi_*(\mathcal{F})$. If \mathcal{F} is a sheaf, we immediately verify the axiom (F) for the presheaf \mathcal{G} , so $\psi_*(\mathcal{F})$ is a sheaf.

(3.4.2). Let $\mathcal{F}_1, \mathcal{F}_2$ be two presheaves of X with values in \mathcal{C} , and let $u : \mathcal{F}_1 \rightarrow \mathcal{F}_2$ be a morphism. When U varies over the set of open subsets of Y , the family of morphisms $u_{\psi^{-1}(U)} : \mathcal{F}_1(\psi^{-1}(U)) \rightarrow \mathcal{F}_2(\psi^{-1}(U))$ satisfies the compatibility conditions with the restriction morphisms, and as a result defines a morphism $\psi_*(u) : \psi_*(\mathcal{F}_1) \rightarrow \psi_*(\mathcal{F}_2)$. If $v : \mathcal{F}_2 \rightarrow \mathcal{F}_3$ is a morphism from \mathcal{F}_2 to a third presheaf on X with values in \mathcal{C} , we have $\psi_*(v \circ u) = \psi_*(v) \circ \psi_*(u)$; in other words, $\psi_*(\mathcal{F})$ is a *covariant functor* in \mathcal{F} , from the category of presheaves (resp. sheaves) on X with values in \mathcal{C} , to that of presheaves (resp. sheaves) on Y with values in \mathcal{C} .

(3.4.3). Let Z be a third topological space, $\psi' : Y \rightarrow Z$ a continuous map, and let $\psi'' = \psi' \circ \psi$. It is clear that we have $\psi''_*(\mathcal{F}) = \psi'_*(\psi_*(\mathcal{F}))$ for each presheaf \mathcal{F} on X with values in \mathcal{C} ; in addition, for each morphism $u : \mathcal{F} \rightarrow \mathcal{G}$ of such presheaves, we have $\psi''_*(u) = \psi'_*(\psi_*(u))$. In other words, ψ''_* is the *composition* of the functors ψ'_* and ψ_* , and this can be written as

$$(\psi' \circ \psi)_* = \psi'_* \circ \psi_*$$

In addition, for each open set U of Y , the image under the restriction $\psi|_{\psi^{-1}(U)}$ of the induced presheaf $\mathcal{F}|_{\psi^{-1}(U)}$ is none other than the induced presheaf $\psi_*(\mathcal{F})|_U$.

(3.4.4). Suppose that the category \mathcal{C} admits inductive limits, and let \mathcal{F} be a presheaf on X with values in \mathcal{C} ; for all $x \in X$, the morphisms $\Gamma(\psi^{-1}(U), \mathcal{F}) \rightarrow \mathcal{F}_x$ (U an open neighbourhood of $\psi(x)$ in Y) form an inductive limit, which gives by passing to the limit a morphism $\psi_x : (\psi_*(\mathcal{F}))_{\psi(x)} \rightarrow \mathcal{F}_x$ of the stalks; in general, these morphisms are *neither injective or surjective*. It is functorial; indeed, if $u : \mathcal{F}_1 \rightarrow \mathcal{F}_2$ is a morphism of presheaves on X with values in \mathcal{C} , the diagram

$$\begin{array}{ccc} (\psi_*(\mathcal{F}_1))_{\psi(x)} & \xrightarrow{\psi_x} & (\mathcal{F}_1)_x \\ (\psi_*(u))_{\psi(x)} \downarrow & & \downarrow u_x \\ (\psi_*(\mathcal{F}_2))_{\psi(x)} & \xrightarrow{\psi_x} & (\mathcal{F}_2)_x \end{array}$$

is commutative. If Z is a third topological space, $\psi' : Y \rightarrow Z$ a continuous map, and $\psi'' = \psi' \circ \psi$, then we have $\psi''_x = \psi_x \circ \psi'_{\psi(x)}$ for $x \in X$.

(3.4.5). Under the hypotheses of (3.4.4), suppose in addition that ψ is a *homeomorphism* from X to the subspace $\psi(X)$ of Y . Then, for each $x \in X$, ψ_x is an *isomorphism*. This applies in particular to the canonical injection j of a subset X of Y into Y .

(3.4.6). Suppose that \mathcal{C} be the category of groups, or of rings, etc. If \mathcal{F} is a sheaf on X with values in \mathcal{C} , of support S , and if $y \notin \overline{\psi(S)}$, then it follows from the definition of $\psi_*(\mathcal{F})$ that $(\psi_*(\mathcal{F}))_y = \{0\}$, or in other words, that the support of $\psi_*(\mathcal{F})$ is contained in $\overline{\psi(S)}$; but it is not necessarily contained in $\psi(S)$. Under the same hypotheses, if j is the canonical injection of a subset X of Y into Y , the sheaf $j_*(\mathcal{F})$ induces \mathcal{F} on X ; if moreover X is *closed* in Y , $j_*(\mathcal{F})$ is the sheaf on Y which induces \mathcal{F} on X and 0 on $Y - X$ (G, II, 2.9.2), but it is in general distinct from the latter when we suppose that X is locally closed but not closed.

3.5. Inverse images of presheaves

(3.5.1). Under the hypotheses of (3.4.1), if \mathcal{F} (resp. \mathcal{G}) is a presheaf on X (resp. Y) with values in \mathbb{C} , then each morphism $u : \mathcal{G} \rightarrow \psi_*(\mathcal{F})$ of presheaves on Y is called a ψ -morphism from \mathcal{G} to \mathcal{F} , and we denote it also by $\mathcal{G} \rightarrow \mathcal{F}$. We denote also by $\text{Hom}_\psi(\mathcal{G}, \mathcal{F})$ the set of $\text{Hom}_Y(\mathcal{G}, \psi_*(\mathcal{F}))$ the ψ -morphisms from \mathcal{G} to \mathcal{F} . For each pair (U, V) , where U is an open set of X , V an open set of Y such that $\psi(U) \subset V$, we have a morphism $u_{U,V} : \mathcal{G}(V) \rightarrow \mathcal{F}(U)$ by composing the restriction morphism $\mathcal{F}(\psi^{-1}(V)) \rightarrow \mathcal{F}(U)$ and the morphism $u_V : \mathcal{G}(V) \rightarrow \psi_*(\mathcal{F})(V) = \mathcal{F}(\psi^{-1}(V))$; it is immediate that these morphisms render commutative the diagrams

$$(3.5.1.1) \quad \begin{array}{ccc} \mathcal{G}(V) & \xrightarrow{u_{U,V}} & \mathcal{F}(U) \\ \downarrow & & \downarrow \\ \mathcal{G}(V') & \xrightarrow{u_{U',V'}} & \mathcal{F}(U') \end{array}$$

for $U' \subset U$, $V' \subset V$, $\psi(U') \subset V'$. Conversely, the data of a family $(u_{U,V})$ of morphisms rendering commutative the diagrams (3.5.1.1) define a ψ -morphism u , since it suffices to take $u_V = u_{\psi^{-1}(V),V}$.

If the category \mathbb{C} admits (generalized) projective limits, and if $\mathfrak{B}, \mathfrak{B}'$ are bases for the topologies of X and Y respectively, to define a ψ -morphism u of sheaves, we can restrict to giving the $u_{U,V}$ for $U \in \mathfrak{B}$, $V \in \mathfrak{B}'$, and $\psi(U) \subset V$, satisfying the compatibility conditions of (3.5.1.1) for U, U' in \mathfrak{B} and V, V' in \mathfrak{B}' ; it indeed suffices to define u_W , for each open $W \subset Y$, as the projective limit of the $u_{U,V}$ for $V \in \mathfrak{B}'$ and $V \subset W$, $U \in \mathfrak{B}$ and $\psi(U) \subset V$.

When the category \mathbb{C} admits inductive limits, we have, for each $x \in X$, a morphism $\mathcal{G}(V) \rightarrow \mathcal{F}(\psi^{-1}(V)) \rightarrow \mathcal{F}_x$, for each open neighbourhood V of $\psi(x)$ in Y , and these morphisms form an inductive system which gives by passing to the limit a morphism $\mathcal{G}_{\psi(x)} \rightarrow \mathcal{F}_x$.

(3.5.2). Under the hypotheses of (3.4.3), let $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be presheaves with values in \mathbb{C} on X, Y, Z respectively, and let $u : \mathcal{G} \rightarrow \psi_*(\mathcal{F})$, $v : \mathcal{H} \rightarrow \psi'_*(\mathcal{G})$ be a ψ -morphism and a ψ' -morphism respectively. We obtain a ψ'' -morphism $w : \mathcal{H} \xrightarrow{v} \psi'_*(\mathcal{G}) \xrightarrow{\psi'_*(u)} \psi'_*(\psi_*(\mathcal{F})) = \psi''_*(\mathcal{F})$, that we call, by definition, the *composition* of u and v . We can therefore consider the pairs (X, \mathcal{F}) consisting of a topological space X and a presheaf \mathcal{F} on X (with values in \mathbb{C}) as forming a *category*, the morphisms being the pairs $(\psi, \theta) : (X, \mathcal{F}) \rightarrow (Y, \mathcal{G})$ consisting of a continuous map $\psi : X \rightarrow Y$ and of a ψ -morphism $\theta : \mathcal{G} \rightarrow \mathcal{F}$.

(3.5.3). Let $\psi : X \rightarrow Y$ be a continuous map, \mathcal{G} a presheaf on Y with values in \mathbb{C} . We call the *inverse image of \mathcal{G} under ψ* the pair (\mathcal{G}', ρ) , where \mathcal{G}' is a sheaf on X with values in \mathbb{C} , and $\rho : \mathcal{G} \rightarrow \mathcal{G}'$ a ψ -morphism (in other words a homomorphism $\mathcal{G} \rightarrow \psi_*(\mathcal{G}')$) such that, for each sheaf \mathcal{F} on X with values in \mathbb{C} , the map

$$(3.5.3.1) \quad \text{Hom}_X(\mathcal{G}', \mathcal{F}) \longrightarrow \text{Hom}_\psi(\mathcal{G}, \mathcal{F}) \longrightarrow \text{Hom}_Y(\mathcal{G}, \psi_*(\mathcal{F}))$$

sending v to $\psi_*(v) \circ \rho$, is a *bijection*; this map, being functorial in \mathcal{F} , then defines an isomorphism of functors in \mathcal{F} . The pair (\mathcal{G}', ρ) is the solution of a universal problem, and we say it is *determined up to unique isomorphism* when it exists. We then write $\mathcal{G}' = \psi^*(\mathcal{G})$, $\rho = \rho_{\mathcal{G}}$, and by abuse of language, we say that $\psi^*(\mathcal{G})$ is the *inverse image sheaf* of \mathcal{G} under ψ , and we agree that $\psi^*(\mathcal{G})$ is considered as equipped with a *canonical ψ -morphism* $\rho_{\mathcal{G}} : \mathcal{G} \rightarrow \psi^*(\mathcal{G})$, that is to say the *canonical homomorphism* of presheaves on Y :

$$(3.5.3.2) \quad \rho_{\mathcal{G}} : \mathcal{G} \longrightarrow \psi^*(\mathcal{G}).$$

For each homomorphism $v : \psi^*(\mathcal{G}) \rightarrow \mathcal{F}$ (where \mathcal{F} is a sheaf on X with values in \mathbb{C}), we put $v^\flat = \psi_*(v) \circ \rho_{\mathcal{G}} : \mathcal{G} \rightarrow \psi_*(\mathcal{F})$. By definition, each morphism of presheaves $u : \mathcal{G} \rightarrow \psi_*(\mathcal{F})$ is of the form v^\flat for a unique v , which we will denote u^\sharp . In other words, each morphism $u : \mathcal{G} \rightarrow \psi_*(\mathcal{F})$ of presheaves factorizes in a unique way as

$$(3.5.3.3) \quad u : \mathcal{G} \xrightarrow{\rho_{\mathcal{G}}} \psi^*(\mathcal{G}) \xrightarrow{\psi^*(u^\sharp)} \psi_*(\mathcal{F}).$$

(3.5.4). Suppose now that the category \mathbf{C} be such⁷ that each presheaf \mathcal{F} on Y with values in \mathbf{C} admits an inverse image under ψ , and we denote it by $\psi^*(\mathcal{F})$.

We will see that we can define $\psi^*(\mathcal{G})$ as a *covariant functor* in \mathcal{G} , from the category of presheaves on Y with values in \mathbf{C} , to that of sheaves on X with values in \mathbf{C} , in such a way that the isomorphism $v \mapsto v^\flat$ is an *isomorphism of bifunctors*

$$(3.5.4.1) \quad \text{Hom}_X(\psi^*(\mathcal{G}), \mathcal{F}) \simeq \text{Hom}_Y(\mathcal{G}, \psi_*(\mathcal{F}))$$

in \mathcal{G} and \mathcal{F} .

Indeed, for each morphism $w : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ of presheaves on Y with values in \mathbf{C} , consider the composite morphism $\mathcal{G}_1 \xrightarrow{w} \mathcal{G}_2 \xrightarrow{\rho_{\mathcal{G}_2}} \psi_*(\psi^*(\mathcal{G}_2))$; to it corresponds a morphism $(\rho_{\mathcal{G}_2} \circ w)^\sharp : \psi^*(\mathcal{G}_1) \rightarrow \psi^*(\mathcal{G}_2)$, that we denote by $\psi^*(w)$. We therefore have, according to (3.5.3.3),

$$(3.5.4.2) \quad \psi_*(\psi^*(w)) \circ \rho_{\mathcal{G}_1} = \rho_{\mathcal{G}_2} \circ w.$$

For each morphism $u : \mathcal{G}_2 \rightarrow \psi_*(\mathcal{F})$, where \mathcal{F} is a sheaf on X with values in \mathbf{C} , we have, according to (3.5.3.3), (3.5.4.2), and the definition of u^\flat , that

$$(u^\flat \circ \psi^*(w))^\flat = \psi_*(u^\flat) \circ \psi_*(\psi^*(w)) \circ \rho_{\mathcal{G}_1} = \psi_*(u^\flat) \circ \rho_{\mathcal{G}_2} \circ w = u \circ w$$

where again

$$(3.5.4.3) \quad (u \circ w)^\sharp = u^\sharp \circ \psi^*(w).$$

If we take in particular for u a morphism $\mathcal{G}_2 \xrightarrow{w'} \mathcal{G}_3 \xrightarrow{\rho_{\mathcal{G}_3}} \psi_*(\psi^*(\mathcal{G}_3))$, it becomes $\psi^*(w' \circ w) = (\rho_{\mathcal{G}_3} \circ w' \circ w)^\sharp = (\rho_{\mathcal{G}_3} \circ w')^\sharp \circ \psi^*(w) = \psi^*(w') \circ \psi^*(w)$, hence our assertion.

Finally, for each sheaf \mathcal{F} on X with values in \mathbf{C} , let $i_{\mathcal{F}}$ be the identity morphism of $\psi_*(\mathcal{F})$ and denote by

$$\sigma_{\mathcal{F}} : \psi^*(\psi_*(\mathcal{F})) \longrightarrow \mathcal{F}$$

the morphism $(i_{\mathcal{F}})^\sharp$; the formula (3.5.4.3) gives in particular the factorization

$$(3.5.4.4) \quad u^\sharp : \psi^*(\mathcal{G}) \xrightarrow{\psi^*(u)} \psi^*(\psi_*(\mathcal{F})) \xrightarrow{\sigma_{\mathcal{F}}} \mathcal{F}$$

for each morphism $u : \mathcal{G} \rightarrow \psi_*(\mathcal{F})$. We say that the morphism $\sigma_{\mathcal{F}}$ is *canonical*.

(3.5.5). Let $\psi' : Y \rightarrow Z$ be a continuous map, and suppose that each presheaf \mathcal{H} on Z with values in \mathbf{C} admits an inverse image $\psi'^*(\mathcal{H})$ under ψ' . Then (with the hypotheses of (3.5.4)) each presheaf \mathcal{H} on Z with values in \mathbf{C} admits an inverse image under $\psi'' = \psi' \circ \psi$ and we have a canonical functorial isomorphism

$$(3.5.5.1) \quad \psi''^*(\mathcal{H}) \simeq \psi^*(\psi'^*(\mathcal{H})).$$

This indeed follows immediately from the definitions, taking into account that $\psi''^* = \psi'^* \circ \psi^*$. In addition, if $u : \mathcal{G} \rightarrow \psi_*(\mathcal{F})$ is a ψ -morphism, $v : \mathcal{H} \rightarrow \psi'_*(\mathcal{G})$ a ψ' -morphism, and $w = \psi'^*(u) \circ v$ their composition (3.5.2), then we have immediately that w^\sharp is the composite morphism

$$w^\sharp : \psi^*(\psi'^*(\mathcal{H})) \xrightarrow{\psi^*(v^\sharp)} \psi^*(\mathcal{G}) \xrightarrow{u^\sharp} \mathcal{F}.$$

(3.5.6). We take in particular for ψ the identity map $1_X : X \rightarrow X$. Then if the inverse image under ψ of a presheaf \mathcal{F} on X with values in \mathbf{C} exists, we say that this inverse image is the *sheaf associated to the presheaf* \mathcal{F} . Each morphism $u : \mathcal{F} \rightarrow \mathcal{F}'$ from \mathcal{F} to a sheaf \mathcal{F}' with values in \mathbf{C} factorizes in a unique way as $\mathcal{F} \xrightarrow{\rho_{\mathcal{F}}} 1_X^*(\mathcal{F}) \xrightarrow{u^\sharp} \mathcal{F}'$.

⁷In the book mentioned in the introduction, we will give very general conditions on the category \mathbf{C} ensuring the existence of inverse images of presheaves with values in \mathbf{C} .

3.6. Simple and locally simple sheaves

(3.6.1). We say that a *presheaf* \mathcal{F} on X , with values in \mathcal{C} , is *constant* if the canonical morphisms $\mathcal{F}(X) \rightarrow \mathcal{F}(U)$ are *isomorphisms* for each nonempty open $U \subset X$; we note that \mathcal{F} is not necessarily a sheaf. We say that a *sheaf* is *simple* if it is the associated sheaf (3.5.6) of a constant presheaf. We say that a sheaf \mathcal{F} is *locally simple* if each $x \in X$ admits an open neighbourhood U such that $\mathcal{F}|_U$ is simple.

(3.6.2). Suppose that X is *irreducible* (2.1.1); then the following properties are equivalent:

- (a) \mathcal{F} is a constant presheaf on X ;
- (b) \mathcal{F} is a simple sheaf on X ;
- (c) \mathcal{F} is a locally simple sheaf on X .

Indeed, let \mathcal{F} be a constant presheaf on X ; if U, V are two nonempty open sets in X , then $U \cap V$ is nonempty, so $\mathcal{F}(X) \rightarrow \mathcal{F}(U) \rightarrow \mathcal{F}(U \cap V)$ and $\mathcal{F}(X) \rightarrow \mathcal{F}(U)$ are isomorphisms, and similarly both $\mathcal{F}(U) \rightarrow \mathcal{F}(U \cap V)$ and $\mathcal{F}(V) \rightarrow \mathcal{F}(U \cap V)$ are isomorphisms. We therefore conclude immediately that the axiom (F) of (3.1.2) is clearly satisfied, \mathcal{F} is isomorphic to its associated sheaf, and as a result (a) implies (b).

Now let (U_α) be an open cover of X by nonempty open sets and let \mathcal{F} be a sheaf on X such that $\mathcal{F}|_{U_\alpha}$ is simple for each α ; as U_α is irreducible, $\mathcal{F}|_{U_\alpha}$ is a constant presheaf according to the above. As $U_\alpha \cap U_\beta$ is not empty, $\mathcal{F}(U_\alpha) \rightarrow \mathcal{F}(U_\alpha \cap U_\beta)$ and $\mathcal{F}(U_\beta) \rightarrow \mathcal{F}(U_\alpha \cap U_\beta)$ are isomorphisms, hence we have a canonical isomorphism $\theta_{\alpha\beta} : \mathcal{F}(U_\alpha) \rightarrow \mathcal{F}(U_\beta)$ for each pair of indices. But then if we apply the condition (F) for $U = X$, we see that for each index α_0 , $\mathcal{F}(U_{\alpha_0})$ and the $\theta_{\alpha_0\alpha}$ are solutions to the universal problem, which (according to the uniqueness) implies that $\mathcal{F}(X) \rightarrow \mathcal{F}(U_{\alpha_0})$ is an isomorphism, and hence proves that (c) implies (a).

3.7. Inverse images of presheaves of groups or rings

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(3.7.1). We will show that when we take \mathcal{C} to be the category of sets, the inverse image under ψ for each presheaf \mathcal{G} with values in \mathcal{C} *always exists* (the notation and hypotheses on X, Y, ψ being that of (3.5.3)). Indeed, for each open $U \subset X$, define $\mathcal{G}'(U)$ as follows: an element s' of $\mathcal{G}'(U)$ is a family $(s'_x)_{x \in U}$, where $s'_x \in \mathcal{G}_{\psi(x)}$ for each $x \in U$, and where, for each $x \in U$, the following condition is satisfied: there exists an open neighbourhood V of $\psi(x)$ in Y , a neighbourhood $W \subset \psi^{-1}(V) \cap U$ of x , and an element $s \in \mathcal{G}(V)$ such that $s'_z = s_{\psi(x)}$ for all $z \in W$. We verify immediately that $U \mapsto \mathcal{G}'(U)$ clearly satisfies the axioms of a *sheaf*.

Now let \mathcal{F} be a sheaf of sets on X , and let $u : \mathcal{G} \rightarrow \psi_*(\mathcal{F}), v : \mathcal{G}' \rightarrow \mathcal{F}$ be morphisms. We define u^\sharp and v^\flat in the following manner: if s' is a section of \mathcal{G}' over a neighbourhood U of $x \in X$ and if V is an open neighbourhood of $\psi(x)$ and $s \in \mathcal{G}(V)$ such that we have $s'_z = s_{\psi(x)}$ for z in a neighbourhood of x contained in $\psi^{-1}(V) \cap U$, we take $u^\sharp_x(s'_x) = u_{\psi(x)}(s_{\psi(x)})$. Similarly, if $s \in \mathcal{G}(V)$ (V open in Y), $v^\flat(s)$ is the section of \mathcal{F} over $\psi^{-1}(V)$, the image under v of the section s' of \mathcal{G}' such that $s'_x = s_{\psi(x)}$ for all $x \in \psi^{-1}(V)$. In addition, the canonical homomorphism (3.5.3) $\rho : \mathcal{G} \rightarrow \psi_*(\psi^*(\mathcal{G}))$ is defined in the following manner: for each open $V \subset Y$ and each section $s \in \Gamma(V, \mathcal{G})$, $\rho(s)$ is the section $(s_{\psi(x)})_{x \in \psi^{-1}(V)}$ of $\psi^*(\mathcal{G})$ over $\psi^{-1}(V)$. The verification of the relations $(u^\sharp)^\flat = u$, $(v^\flat)^\sharp = v$, and $v^\flat = \psi_*(v) \circ \rho$ is immediate, and proves our assertion.

We check that, if $w : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ is a homomorphism of sheaves of sets on Y , $\psi^*(w)$ is expressed in the following manner: if $s' = (s'_x)_{x \in U}$ is a section of $\psi^*(\mathcal{G}_1)$ over an open set U of X , then $(\psi^*(w))(s')$ is the family $(w_{\psi(x)}(s'_x))_{x \in U}$. Finally, it is immediate that for each open set V of Y , the inverse image of $\mathcal{G}|_V$ under the restriction of ψ to $\psi^{-1}(V)$ is identical to the induced sheaf $\psi^*(\mathcal{G})|_{\psi^{-1}(V)}$.

When ψ is the identity 1_X , we recover the definition of a sheaf of sets associated to a presheaf (G, II, 1.2). The above considerations apply without change when \mathcal{C} is the category of groups or of rings (not necessarily commutative).

When X is any subset of a topological space Y , and j the canonical injection $X \rightarrow Y$, for each sheaf \mathcal{G} on Y with values in a category \mathcal{C} , we call the *induced* sheaf of X by \mathcal{G} the inverse image $j^*(\mathcal{G})$ (whenever it exists); for the sheaves of sets (or of groups, or of rings) we recover the usual definition (G, II, 1.5).

(3.7.2). Keeping the notation and hypotheses of (3.5.3), suppose that \mathcal{G} is a *sheaf* of groups (resp. of rings) on Y . The definition of sections of $\psi^*(\mathcal{G})$ (3.7.1) shows (taking into account (3.4.4)) that the homomorphism of stalks $\psi_x \circ \rho_{\psi(x)} : \mathcal{G}_{\psi(x)} \rightarrow (\psi^*(\mathcal{G}))_x$ is a *functorial isomorphism in \mathcal{G}* , that identifies the two stalks; with this identification, u_x^\sharp is identical to the homomorphism defined in (3.5.1), and in particular, we have $\text{Supp}(\psi^*(\mathcal{G})) = \psi^{-1}(\text{Supp}(\mathcal{G}))$.

An immediate consequence of this result is that *the functor $\psi^*(\mathcal{G})$ is exact in \mathcal{G} on the abelian category of sheaves of abelian groups.*

3.8. Sheaves on pseudo-discrete spaces

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(3.8.1). Let X be a topological space whose topology admits a basis \mathfrak{B} consisting of open *quasi-compact* subsets. Let \mathcal{F} be a *sheaf of sets* on X ; if we equip each of the $\mathcal{F}(U)$ with the *discrete* topology, $U \mapsto \mathcal{F}(U)$ is a *presheaf of topological spaces*. We will see that there exists a *sheaf of topological spaces \mathcal{F}' associated to \mathcal{F}* (3.5.6) such that $\Gamma(U, \mathcal{F}')$ is the discrete space $\mathcal{F}(U)$ for each open *quasi-compact* subsets U . It will suffice to show that the presheaf $U \mapsto \mathcal{F}(U)$ of discrete topological spaces on \mathfrak{B} satisfy the condition (F₀) of (3.2.2), and more generally that if U is an open quasi-compact subset and if (U_α) is a cover of U by sets of \mathfrak{B} , then the least fine topology \mathcal{T} on $\Gamma(U, \mathcal{F})$ renders continuous the maps $\Gamma(U, \mathcal{F}) \rightarrow \Gamma(U_\alpha, \mathcal{F})$ is the *discrete* topology. There exists a finite number of indices α_i such that $U = \bigcup_i U_{\alpha_i}$. Let $s \in \Gamma(U, \mathcal{F})$ and let s_i be its image in $\Gamma(U_{\alpha_i}, \mathcal{F})$; the intersection of the inverse images of the sets $\{s_i\}$ is by definition a neighbourhood of s for \mathcal{T} ; but since \mathcal{F} is a sheaf of sets and the U_{α_i} cover U , this intersection is reduced to s , hence our assertion.

We note that if U is an open non quasi-compact subset of X , the topological space $\Gamma(U, \mathcal{F}')$ still has $\Gamma(U, \mathcal{F})$ as the underlying set, but the topology is not discrete in general: it is the least fine rendering commutative the maps $\Gamma(U, \mathcal{F}) \rightarrow \Gamma(V, \mathcal{F})$, for $V \in \mathfrak{B}$ and $V \subset U$ (the $\Gamma(V, \mathcal{F})$ being discrete).

The above considerations apply without modification to sheaves of groups or of rings (not necessarily commutative), and associate to them sheaves of *topological groups* or *topological rings*, respectively. To summarize, we say that the sheaf \mathcal{F}' is the *pseudo-discrete* sheaf of *spaces* (resp. *groups, rings*) associated to a sheaf of sets (resp. groups, rings) \mathcal{F} .

(3.8.2). Let \mathcal{F}, \mathcal{G} be two sheaves of sets (resp. groups, rings) on X , $u : \mathcal{F} \rightarrow \mathcal{G}$ a homomorphism. Then u is thus a *continuous* homomorphism $\mathcal{F}' \rightarrow \mathcal{G}'$, if we denote by \mathcal{F}' and \mathcal{G}' the pseudo-discrete sheaves associated to \mathcal{F} and \mathcal{G} ; this follows in effect from (3.2.5).

(3.8.3). Let \mathcal{F} be a sheaf of sets, \mathcal{H} a subsheaf of \mathcal{F} , \mathcal{F}' and \mathcal{H}' the pseudo-discrete sheaves associated to \mathcal{F} and \mathcal{H} respectively. Then, for each open $U \subset X$, $\Gamma(U, \mathcal{H}')$ is *closed* in $\Gamma(U, \mathcal{F}')$: indeed, it is the intersection of the inverse images of the $\Gamma(V, \mathcal{H})$ (for $V \in \mathfrak{B}$, $V \subset U$) under the continuous maps $\Gamma(U, \mathcal{F}) \rightarrow \Gamma(V, \mathcal{F})$, and $\Gamma(V, \mathcal{H})$ is closed in the discrete space $\Gamma(V, \mathcal{F})$.

§4. RINGED SPACES

4.1. Ringed spaces, sheaves of \mathcal{A} -modules, \mathcal{A} -algebras

(4.1.1). A *ringed space* (resp. topologically ringed space) is a pair (X, \mathcal{A}) consisting of a topological space X and a sheaf of (not necessarily commutative) rings (resp. of a sheaf of topological rings) \mathcal{A} on X ; we say that X is the *underlying* topological space of the ringed space (X, \mathcal{A}) , and \mathcal{A} the *structure sheaf*. The latter is denoted \mathcal{O}_X , and its stalk at a point $x \in X$ is denoted $\mathcal{O}_{X,x}$ or simply \mathcal{O}_x when there is no chance of confusion.

We denote by 1 or e the *unit section* of \mathcal{O}_X over X (the unit element of $\Gamma(X, \mathcal{O}_X)$).

As in this treatise we will have to consider in particular sheaves of *commutative* rings, it will be understood, when we speak of a ringed space (X, \mathcal{A}) without specification, that \mathcal{A} is a sheaf of commutative rings.

The ringed spaces with not-necessarily-commutative structure sheaves (resp. the topologically ringed spaces) form a *category*, where we define a *morphism* $(X, \mathcal{A}) \rightarrow (Y, \mathcal{B})$ as a couple $(\psi, \theta) = \Psi$ consisting of a continuous map $\psi : X \rightarrow Y$ and a ψ -*morphism* $\theta : \mathcal{B} \rightarrow \mathcal{A}$ (3.5.1) of sheaves of rings (resp. of sheaves of topological rings); the *composition* of a second morphism $\Psi' = (\psi', \theta') : (Y, \mathcal{B}) \rightarrow (Z, \mathcal{C})$ and of Ψ , denoted $\Psi'' = \Psi' \circ \Psi$, is the morphism (ψ'', θ'') where $\psi'' = \psi' \circ \psi$, and θ'' is the composition of θ and θ' (equal to $\psi'_*(\theta) \circ \theta'$, cf. (3.5.2)). For ringed spaces, remember that

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we then have $\theta''^\sharp = \theta^\sharp \circ \psi^*(\theta'^\sharp)$ (3.5.5); therefore if θ'^\sharp and θ^\sharp are *injective* (resp. *surjective*), then the same is true of θ''^\sharp , taking into account that $\psi_x \circ \rho_{\psi(x)}$ is an isomorphism for all $x \in X$ (3.7.2). We verify immediately, thanks to the above, that when ψ is an *injective* continuous map and when θ^\sharp is a *surjective* homomorphism of sheaves of rings, the morphism (ψ, θ) is a *monomorphism* (T, 1.1) in the category of ringed spaces.

By abuse of language, we will often replace ψ by Ψ in notation, for example in writing $\Psi^{-1}(U)$ in place of $\psi^{-1}(U)$ for a subset U of Y , when there is no risk of confusion.

(4.1.2). For each subset M of X , the pair $(M, \mathcal{A}|_M)$ is evidently a ringed space, said to be *induced* on M by the ringed space (X, \mathcal{A}) (and is still called the *restriction* of (X, \mathcal{A}) to M). If j is the canonical injection $M \rightarrow X$ and ω is the identity map of $\mathcal{A}|_M$, (j, ω^\flat) is a monomorphism $(M, \mathcal{A}|_M) \rightarrow (X, \mathcal{A})$ of ringed spaces, called the *canonical injection*. The composition of a morphism $\Psi : (X, \mathcal{A}) \rightarrow (Y, \mathcal{B})$ and this injection is called the *restriction* of Ψ to M .

(4.1.3). We will not revisit the definitions of \mathcal{A} -modules or *algebraic sheaves* on a ringed space (X, \mathcal{A}) (G, II, 2.2); when \mathcal{A} is a sheaf of not necessarily commutative rings, by \mathcal{A} -module we will always mean “left \mathcal{A} -module” unless expressly stated otherwise. The \mathcal{A} -submodules of \mathcal{A} will be called *sheaves of ideals* (left, right, or two-sided) in \mathcal{A} or \mathcal{A} -ideals.

When \mathcal{A} is a sheaf of commutative rings, and in the definition of \mathcal{A} -modules, we replace everywhere the *module* structure by that of an *algebra*, we obtain the definition of an \mathcal{A} -algebra on X . It is the same to say that an \mathcal{A} -algebra (not necessarily commutative) is a \mathcal{A} -module \mathcal{C} , given with a homomorphism of \mathcal{A} -modules $\phi : \mathcal{C} \otimes_{\mathcal{A}} \mathcal{C} \rightarrow \mathcal{C}$ and a section e over X , such that: 1st the diagram

$$\begin{array}{ccc} \mathcal{C} \otimes_{\mathcal{A}} \mathcal{C} \otimes_{\mathcal{A}} \mathcal{C} & \xrightarrow{\phi \otimes 1} & \mathcal{C} \otimes_{\mathcal{A}} \mathcal{C} \\ \downarrow 1 \otimes \phi & & \downarrow \phi \\ \mathcal{C} \otimes_{\mathcal{A}} \mathcal{C} & \xrightarrow{\phi} & \mathcal{C} \end{array}$$

is commutative; 2nd for each open $U \subset X$ and each section $s \in \Gamma(U, \mathcal{C})$, we have $\phi((e|U) \otimes s) = \phi(s \otimes (e|U)) = s$. We say that \mathcal{C} is a commutative \mathcal{A} -algebra if the diagram

$$\begin{array}{ccc} \mathcal{C} \otimes_{\mathcal{A}} \mathcal{C} & \xrightarrow{\sigma} & \mathcal{C} \otimes_{\mathcal{A}} \mathcal{C} \\ & \searrow \phi & \swarrow \phi \\ & \mathcal{C} & \end{array}$$

is commutative, σ denoting the canonical symmetry (twist) map of the tensor product $\mathcal{C} \otimes_{\mathcal{A}} \mathcal{C}$.

The homomorphisms of \mathcal{A} -algebras are also defined as the homomorphisms of \mathcal{A} -modules in (G, II, 2.2), but naturally no longer form an abelian group.

If \mathcal{M} is an \mathcal{A} -submodule of an \mathcal{A} -algebra \mathcal{C} , the \mathcal{A} -subalgebra of \mathcal{C} generated by \mathcal{M} is the sum of the images of the homomorphisms $\otimes^n \mathcal{M} \rightarrow \mathcal{C}$ (for each $n \geq 0$). This is also the sheaf associated to the presheaf $U \mapsto \mathcal{B}(U)$ of algebras, $\mathcal{B}(U)$ being the subalgebra of $\Gamma(U, \mathcal{C})$ generated by the submodule $\Gamma(U, \mathcal{M})$.

(4.1.4). We say that a sheaf of rings \mathcal{A} on a topological space X is *reduced at a point x in X* if the stalk \mathcal{A}_x is a *reduced ring* (1.1.1); we say that \mathcal{A} is *reduced* if it is reduced at all points of X . Recall that a ring A is called *regular* if each of the local rings $A_{\mathfrak{p}}$ (where \mathfrak{p} varies over the set of prime ideals of A) is a regular local ring; we will say that a sheaf of rings \mathcal{A} on X is *regular at a point x* (resp. *regular*) if the stalk \mathcal{A}_x is a regular ring (resp. if \mathcal{A} is regular at each point). Finally, we will say that a sheaf of rings \mathcal{A} on X is *normal at a point x* (resp. *normal*) if the stalk \mathcal{A}_x is an *integral and integrally closed ring* (resp. if \mathcal{A} is normal at each point). We will say that a ringed space (X, \mathcal{A}) has any of these preceding properties if the sheaf of rings \mathcal{A} has that property.

A *graded* sheaf of rings \mathcal{A} is by definition a sheaf of rings that is the direct sum (G, II, 2.7) of a family $(\mathcal{A}_n)_{n \in \mathbb{Z}}$ of sheaves of abelian groups with the conditions $\mathcal{A}_m \mathcal{A}_n \subset \mathcal{A}_{m+n}$; a *graded \mathcal{A} -module* is an \mathcal{A} -module \mathcal{F} that is the direct sum of a family $(\mathcal{F}_n)_{n \in \mathbb{Z}}$ of sheaves of abelian groups, satisfying the conditions $\mathcal{A}_m \mathcal{F}_n \subset \mathcal{F}_{m+n}$. It is equivalent to say that $(\mathcal{A}_m)_x (\mathcal{A}_n)_x \subset (\mathcal{A}_{m+n})_x$ (resp. $(\mathcal{A}_m)_x (\mathcal{F}_n)_x \subset (\mathcal{F}_{m+n})_x$) for each point x .

(4.1.5). Given a ringed space (X, \mathcal{A}) (not necessarily commutative), we will not recall here the definitions of the bifunctors $\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G}$, $\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{F})$, and $\text{Hom}_{\mathcal{A}}(\mathcal{F}, \mathcal{G})$ (G, II, 2.8 and 2.2) in the categories of left or right (depending on the case) \mathcal{A} -modules, with values in the category of sheaves of abelian groups (or more generally of \mathcal{C} -modules, if \mathcal{C} is the center of \mathcal{A}). The stalk $(\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G})_x$ for each point $x \in X$ canonically identifies with $\mathcal{F}_x \otimes_{\mathcal{A}_x} \mathcal{G}_x$ and we define a canonical and functorial homomorphism $(\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G}))_x \rightarrow \text{Hom}_{\mathcal{A}_x}(\mathcal{F}_x, \mathcal{G}_x)$ which is in general neither injective nor surjective. The bifunctors considered above are additive and in particular, commute with finite direct limits; $\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G}$ is right exact in \mathcal{F} and in \mathcal{G} , commutes with inductive limits, and $\mathcal{A} \otimes_{\mathcal{A}} \mathcal{G}$ (resp. $\mathcal{F} \otimes_{\mathcal{A}} \mathcal{A}$) canonically identifies with \mathcal{G} (resp. \mathcal{F}). The functors $\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G})$ and $\text{Hom}_{\mathcal{A}}(\mathcal{F}, \mathcal{G})$ are *left exact* in \mathcal{F} and \mathcal{G} ; more precisely, if we have an exact sequence of the form $0 \rightarrow \mathcal{G}' \rightarrow \mathcal{G} \rightarrow \mathcal{G}''$, the sequence

$$0 \rightarrow \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G}') \rightarrow \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G}) \rightarrow \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G}'')$$

is exact, and if we have an exact sequence of the form $\mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$, the sequence

$$0 \rightarrow \mathcal{H}om_{\mathcal{A}}(\mathcal{F}'', \mathcal{G}) \rightarrow \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G}) \rightarrow \mathcal{H}om_{\mathcal{A}}(\mathcal{F}', \mathcal{G})$$

is exact, with the analogous properties for the functor Hom . In addition, $\mathcal{H}om_{\mathcal{A}}(\mathcal{A}, \mathcal{G})$ canonically identifies with \mathcal{G} ; finally, for each open $U \subset X$, we have

$$\Gamma(U, \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G})) = \text{Hom}_{\mathcal{A}|U}(\mathcal{F}|U, \mathcal{G}|U).$$

For each left (resp. right) \mathcal{A} -module, we define the *dual* of \mathcal{F} and denote it by \mathcal{F}^\vee the right (resp. left) \mathcal{A} -module $\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{A})$.

Finally, if \mathcal{A} is a sheaf of commutative rings, \mathcal{F} an \mathcal{A} -module, $U \mapsto \wedge^p \Gamma(U, \mathcal{F})$ is a presheaf whose associated sheaf is an \mathcal{A} -module denoted $\wedge^p \mathcal{F}$ and is called the *p-th exterior power* of \mathcal{F} ; we verify easily that the canonical map of the presheaf $U \mapsto \wedge^p \Gamma(U, \mathcal{F})$ to the associated sheaf $\wedge^p \mathcal{F}$ is *injective*, and for each $x \in X$, $(\wedge^p \mathcal{F})_x = \wedge^p(\mathcal{F}_x)$. It is clear that $\wedge^p \mathcal{F}$ is a covariant functor in \mathcal{F} .

(4.1.6). Suppose that \mathcal{A} is a sheaf of not-necessarily-commutative rings, \mathcal{I} a left sheaf of ideals of \mathcal{A} , \mathcal{F} an left \mathcal{A} -module; we then denote by $\mathcal{I}\mathcal{F}$ the \mathcal{A} -submodule of \mathcal{F} , the image of $\mathcal{I} \otimes_{\mathbf{Z}} \mathcal{F}$ (where \mathbf{Z} is the sheaf associated to the constant presheaf $U \mapsto \mathbf{Z}$) under the canonical map $\mathcal{I} \otimes_{\mathbf{Z}} \mathcal{F} \rightarrow \mathcal{F}$; it is clear that for each $x \in X$, we have $(\mathcal{I}\mathcal{F})_x = \mathcal{I}_x \mathcal{F}_x$. When \mathcal{A} is commutative, $\mathcal{I}\mathcal{F}$ is also the canonical image of $\mathcal{I} \otimes_{\mathcal{A}} \mathcal{F} \rightarrow \mathcal{F}$. It is immediate that $\mathcal{I}\mathcal{F}$ is also the \mathcal{A} -module associated to the presheaf $U \mapsto \Gamma(U, \mathcal{I})\Gamma(U, \mathcal{F})$. If $\mathcal{I}_1, \mathcal{I}_2$ are two left sheaves of ideals of \mathcal{A} , we have $\mathcal{I}_1(\mathcal{I}_2\mathcal{F}) = (\mathcal{I}_1\mathcal{I}_2)\mathcal{F}$.

(4.1.7). Let $(X_\lambda, \mathcal{A}_\lambda)_{\lambda \in L}$ be a family of ringed spaces; for each couple (λ, μ) , suppose we are given an open subset $V_{\lambda\mu}$ of X_λ , and an isomorphism of ringed spaces $\phi_{\lambda\mu} : (V_{\mu\lambda}, \mathcal{A}_\mu|_{V_{\lambda\mu}}) \simeq (V_{\lambda\mu}, \mathcal{A}_\lambda|_{V_{\lambda\mu}})$, with $V_{\lambda\lambda} = X_\lambda$, $\phi_{\lambda\lambda}$ being the identity. Furthermore, suppose that, for each triple (λ, μ, ν) , if we denote by $\phi'_{\mu\lambda}$ the restriction of $\phi_{\mu\lambda}$ to $V_{\lambda\mu} \cap V_{\lambda\nu}$, $\phi'_{\mu\lambda}$ is an isomorphism from $(V_{\lambda\mu} \cap V_{\lambda\nu}, \mathcal{A}_\lambda|_{(V_{\lambda\mu} \cap V_{\lambda\nu})})$ to $(V_{\mu\nu} \cap V_{\mu\lambda}, \mathcal{A}_\mu|_{(V_{\mu\nu} \cap V_{\mu\lambda})})$ and that we have $\phi'_{\lambda\nu} = \phi'_{\lambda\mu} \circ \phi'_{\mu\nu}$ (*gluing condition* for the $\phi_{\lambda\mu}$).

We can first consider the topological space obtained by gluing (by means of the $\phi_{\lambda\mu}$) of the X_λ along the $V_{\lambda\mu}$; if we identify X_λ with the corresponding open subset X'_λ in X , the hypotheses imply that the three sets $V_{\lambda\mu} \cap V_{\lambda\nu}$, $V_{\mu\nu} \cap V_{\mu\lambda}$, $V_{\nu\lambda} \cap V_{\nu\mu}$ identify with $X'_\lambda \cap X'_\mu \cap X'_\nu$. We can also transport to X'_λ the ringed space structure of X_λ , and if \mathcal{A}'_λ are the transported sheaves of rings corresponding to the \mathcal{A}_λ , the \mathcal{A}'_λ satisfy the gluing condition (3.3.1) and therefore define a sheaf of rings \mathcal{A} on X ; we say that (X, \mathcal{A}) is the ringed space obtained by *gluing the $(X_\lambda, \mathcal{A}_\lambda)$ along the $V_{\lambda\mu}$* , by means of the $\phi_{\lambda\mu}$.

4.2. Direct image of an \mathcal{A} -module

(4.2.1). Let $(X, \mathcal{A}), (Y, \mathcal{B})$ be two ringed spaces, $\Psi = (\psi, \theta)$ a morphism $(X, \mathcal{A}) \rightarrow (Y, \mathcal{B})$; $\psi_*(\mathcal{A})$ is then a sheaf of rings on Y , and θ a homomorphism $\mathcal{B} \rightarrow \psi_*(\mathcal{A})$ of sheaves of rings. Then let \mathcal{F} be an \mathcal{A} -module; the direct image $\psi_*(\mathcal{F})$ is a sheaf of abelian groups on Y . In addition, for each open $U \subset Y$,

$$\Gamma(U, \psi_*(\mathcal{F})) = \Gamma(\psi^{-1}(U), \mathcal{F})$$

is equipped with the structure of a module over the ring $\Gamma(U, \psi_*(\mathcal{A})) = \Gamma(\psi^{-1}(U), \mathcal{A})$; the bilinear maps which define these structures are compatible with the restriction operations, defining on $\psi_*(\mathcal{F})$ the structure of a $\psi_*(\mathcal{A})$ -module. The homomorphism $\theta : \mathcal{B} \rightarrow \psi_*(\mathcal{A})$ then defines also on $\psi_*(\mathcal{F})$ a \mathcal{B} -module structure; we say that this \mathcal{B} -module is the *direct image of \mathcal{F} under the morphism Ψ* , and we denote it by $\Psi_*(\mathcal{F})$. If $\mathcal{F}_1, \mathcal{F}_2$ are two \mathcal{A} -modules over X and u an \mathcal{A} -homomorphism $\mathcal{F}_1 \rightarrow \mathcal{F}_2$, it is immediate (by considering the sections over the open subsets of Y) that $\psi_*(u)$ is a $\psi_*(\mathcal{A})$ -homomorphism $\psi_*(\mathcal{F}_1) \rightarrow \psi_*(\mathcal{F}_2)$, and *a fortiori* a \mathcal{B} -homomorphism $\Psi_*(\mathcal{F}_1) \rightarrow \Psi_*(\mathcal{F}_2)$; as a \mathcal{B} -homomorphism, we denote it by $\Psi_*(u)$. So we see that Ψ_* is a *covariant functor* from the category of \mathcal{A} -modules to that of \mathcal{B} -modules. In addition, it is immediate that this functor is *left exact* (G, II, 2.12).

On $\psi_*(\mathcal{A})$, the structure of a \mathcal{B} -module and the structure of a sheaf of rings define a \mathcal{B} -algebra structure; we denote by $\Psi_*(\mathcal{A})$ this \mathcal{B} -algebra.

(4.2.2). Let \mathcal{M}, \mathcal{N} be two \mathcal{A} -modules. For each open set U of Y , we have a canonical map

$$\Gamma(\psi^{-1}(U), \mathcal{M}) \times \Gamma(\psi^{-1}(U), \mathcal{N}) \longrightarrow \Gamma(\psi^{-1}(U), \mathcal{M} \otimes_{\mathcal{A}} \mathcal{N})$$

which is bilinear over the ring $\Gamma(\psi^{-1}(U), \mathcal{A}) = \Gamma(U, \psi_*(\mathcal{A}))$, and *a fortiori* over $\Gamma(U, \mathcal{B})$; it therefore defines a homomorphism

$$\Gamma(U, \Psi_*(\mathcal{M})) \otimes_{\Gamma(U, \mathcal{B})} \Gamma(U, \Psi_*(\mathcal{N})) \longrightarrow \Gamma(U, \Psi_*(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}))$$

and as we check immediately that these homomorphisms are compatible with the restriction operations, they give a canonical functorial homomorphism of \mathcal{B} -modules

$$(4.2.2.1) \quad \Psi_*(\mathcal{M}) \otimes_{\mathcal{B}} \Psi_*(\mathcal{N}) \longrightarrow \Psi_*(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N})$$

which is in general neither injective nor surjective. If \mathcal{P} is a third \mathcal{A} -module, we check immediately that the diagram

$$(4.2.2.2) \quad \begin{array}{ccc} \Psi_*(\mathcal{M}) \otimes_{\mathcal{B}} \Psi_*(\mathcal{N}) \otimes_{\mathcal{B}} \Psi_*(\mathcal{P}) & \longrightarrow & \Psi_*(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}) \otimes_{\mathcal{B}} \Psi_*(\mathcal{P}) \\ \downarrow & & \downarrow \\ \Psi_*(\mathcal{M}) \otimes_{\mathcal{B}} \Psi_*(\mathcal{N} \otimes_{\mathcal{A}} \mathcal{P}) & \longrightarrow & \Psi_*(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N} \otimes_{\mathcal{A}} \mathcal{P}) \end{array}$$

is commutative.

(4.2.3). Let \mathcal{M}, \mathcal{N} be two \mathcal{A} -modules. For each open $U \subset Y$, we have by definition that $\Gamma(\psi^{-1}(U), \mathcal{H}om_{\mathcal{A}}(\mathcal{M}, \mathcal{N})) = \mathcal{H}om_{\mathcal{A}|V}(\mathcal{M}|V, \mathcal{N}|V)$, where we put $V = \psi^{-1}(U)$; the map $u \mapsto \Psi_*(u)$ is a homomorphism

$$\mathcal{H}om_{\mathcal{A}|V}(\mathcal{M}|V, \mathcal{N}|V) \longrightarrow \mathcal{H}om_{\mathcal{B}|U}(\Psi_*(\mathcal{M})|U, \Psi_*(\mathcal{N})|U)$$

on the $\Gamma(U, \mathcal{B})$ -module structures; these homomorphisms are compatible with the restriction operations, hence they define a canonical functorial homomorphism of \mathcal{B} -modules

$$(4.2.3.1) \quad \Psi_*(\mathcal{H}om_{\mathcal{A}}(\mathcal{M}, \mathcal{N})) \longrightarrow \mathcal{H}om_{\mathcal{B}}(\Psi_*(\mathcal{M}), \Psi_*(\mathcal{N})).$$

(4.2.4). If \mathcal{C} is an \mathcal{A} -algebra, the composite homomorphism

$$\Psi_*(\mathcal{C}) \otimes_{\mathcal{B}} \Psi_*(\mathcal{C}) \longrightarrow \Psi_*(\mathcal{C} \otimes_{\mathcal{A}} \mathcal{C}) \longrightarrow \Psi_*(\mathcal{C})$$

defines on $\Psi_*(\mathcal{C})$ the structure of a \mathcal{B} -algebra, as a result of (4.2.2.2). We see similarly that if \mathcal{M} is a \mathcal{C} -module, $\Psi_*(\mathcal{M})$ is canonically equipped with the structure of a $\Psi_*(\mathcal{C})$ -module.

(4.2.5). Consider in particular the case where X is a *closed* subspace of Y and where ψ is the canonical injection $j : X \rightarrow Y$. If $\mathcal{B}' = \mathcal{B}|_X = j^*(\mathcal{B})$ is the restriction of the sheaf of rings \mathcal{B} to X , an \mathcal{A} -module \mathcal{M} can be considered as a \mathcal{B}' -module by means of the homomorphism $\theta^\sharp : \mathcal{B}' \rightarrow \mathcal{A}$; then $\Psi_*(\mathcal{M})$ is the \mathcal{B} -module which induces \mathcal{M} on X and 0 elsewhere. If \mathcal{N} is a second \mathcal{A} -module, $\Psi_*(\mathcal{M}) \otimes_{\mathcal{B}} \Psi_*(\mathcal{N})$ canonically identifies with $\Psi_*(\mathcal{M} \otimes_{\mathcal{B}'} \mathcal{N})$ and $\mathcal{H}om_{\mathcal{B}}(\Psi_*(\mathcal{M}), \Psi_*(\mathcal{N}))$ with $\Psi_*(\mathcal{H}om_{\mathcal{B}'}(\mathcal{M}, \mathcal{N}))$.

(4.2.6). Let (Z, \mathcal{C}) be a third ringed space, $\Psi' = (\psi', \theta')$ a morphism $(Y, \mathcal{B}) \rightarrow (Z, \mathcal{C})$; if Ψ'' is the composite morphism $\Psi' \circ \Psi$, it is clear that we have $\Psi''_* = \Psi'_* \circ \Psi_*$.

4.3. Inverse image of an \mathcal{A} -module

(4.3.1). The hypotheses and notation being the same as (4.2.1), let \mathcal{G} be a \mathcal{B} -module and $\psi^*(\mathcal{G})$ the inverse image (3.7.1) which is therefore a sheaf of abelian groups on X . The definition of sections of $\psi^*(\mathcal{G})$ and of $\psi^*(\mathcal{B})$ (3.7.1) shows that $\psi^*(\mathcal{G})$ is canonically equipped with a $\psi^*(\mathcal{B})$ -module structure. On the other hand, the homomorphism $\theta^\sharp : \psi^*(\mathcal{B}) \rightarrow \mathcal{A}$ endows \mathcal{A} with the a $\psi^*(\mathcal{B})$ -module structure, which we denote by $\mathcal{A}_{[\theta]}$ when necessary to avoid confusion; the tensor product $\psi^*(\mathcal{G}) \otimes_{\psi^*(\mathcal{B})} \mathcal{A}_{[\theta]}$ is then equipped with an \mathcal{A} -module structure. We say that this \mathcal{A} -module is *the inverse image of \mathcal{G} under the morphism Ψ* and we denote it by $\Psi^*(\mathcal{G})$. If $\mathcal{G}_1, \mathcal{G}_2$ are two \mathcal{B} -modules over Y , v a \mathcal{B} -homomorphism $\mathcal{G}_1 \rightarrow \mathcal{G}_2$, then $\psi^*(v)$, as we check immediately, is a $\psi^*(\mathcal{B})$ -homomorphism from $\psi^*(\mathcal{G}_1)$ to $\psi^*(\mathcal{G}_2)$; as a result $\psi^*(v) \otimes 1$ is an \mathcal{A} -homomorphism $\Psi^*(\mathcal{G}_1) \rightarrow \Psi^*(\mathcal{G}_2)$, which we denote by $\Psi^*(v)$. So we define Ψ^* as a *covariant functor* from the category of \mathcal{B} -modules to that of \mathcal{A} -modules. Here, this functor (contrary to ψ^*) is no longer exact in general, but only *right exact*, the tensorization by \mathcal{A} being a right exact functor to the category of $\psi^*(\mathcal{B})$ -modules.

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For each $x \in X$, we have $(\Psi^*(\mathcal{G}))_x = \mathcal{G}_{\psi(x)} \otimes_{\mathcal{B}_{\psi(x)}} \mathcal{A}_x$, according to (3.7.2). The support of $\Psi^*(\mathcal{G})$ is thus contained in $\psi^{-1}(\text{Supp}(\mathcal{G}))$.

(4.3.2). Let (\mathcal{G}_λ) be an inductive system of \mathcal{B} -modules, and let $\mathcal{G} = \varinjlim \mathcal{G}_\lambda$ be its inductive limit. The canonical homomorphisms $\mathcal{G}_\lambda \rightarrow \mathcal{G}$ define the $\psi^*(\mathcal{B})$ -homomorphisms $\psi^*(\mathcal{G}_\lambda) \rightarrow \psi^*(\mathcal{G})$, which give a canonical homomorphism $\varinjlim \psi^*(\mathcal{G}_\lambda) \rightarrow \psi^*(\mathcal{G})$. As the stalk at a point of an inductive limit of sheaves is the inductive limit of the stalks at the same point (G, II, 1.11), the preceding canonical homomorphism is *bijective* (3.7.2). In addition, the tensor product commutes with inductive limits of sheaves, and we thus have a *canonical functorial isomorphism* $\varinjlim \Psi^*(\mathcal{G}_\lambda) \simeq \Psi^*(\varinjlim \mathcal{G}_\lambda)$ of \mathcal{A} -modules.

On the other hand, for a finite direct sum $\bigoplus_i \mathcal{G}_i$ of \mathcal{B} -modules, it is clear that $\psi^*(\bigoplus_i \mathcal{G}_i) = \bigoplus_i \psi^*(\mathcal{G}_i)$, therefore, by tensoring with $\mathcal{A}_{[\theta]}$,

$$(4.3.2.1) \quad \Psi^*\left(\bigoplus_i \mathcal{G}_i\right) = \bigoplus_i \Psi^*(\mathcal{G}_i).$$

By passing to the inductive limit, we deduce, in light of the above, that the above equality is still true for *any* direct sum.

(4.3.3). Let $\mathcal{G}_1, \mathcal{G}_2$ be two \mathcal{B} -modules; from the definition of the inverse images of sheaves of abelian groups (3.7.1), we obtain immediately a canonical homomorphism $\psi^*(\mathcal{G}_1) \otimes_{\psi^*(\mathcal{B})} \psi^*(\mathcal{G}_2) \rightarrow \psi^*(\mathcal{G}_1 \otimes_{\mathcal{B}} \mathcal{G}_2)$ of $\psi^*(\mathcal{B})$ -modules, and the stalk at a point of a tensor product of sheaves being the tensor product of the stalks at this point (G, II, 2.8), we deduce from (3.7.2) that the above homomorphism is in fact a *isomorphism*. By tensoring with \mathcal{A} , we obtain a *canonical functorial isomorphism*

$$(4.3.3.1) \quad \Psi^*(\mathcal{G}_1) \otimes_{\mathcal{A}} \Psi^*(\mathcal{G}_2) \simeq \Psi^*(\mathcal{G}_1 \otimes_{\mathcal{B}} \mathcal{G}_2).$$

(4.3.4). Let \mathcal{C} be a \mathcal{B} -algebra; the data of the algebra structure on \mathcal{C} is the same as the data of a \mathcal{B} -homomorphism $\mathcal{C} \otimes_{\mathcal{B}} \mathcal{C} \rightarrow \mathcal{C}$ satisfying the associativity and commutativity conditions (conditions which are checked stalk-wise); the above isomorphism allows us to consider this homomorphism as a homomorphism of \mathcal{A} -modules $\Psi^*(\mathcal{C}) \otimes_{\mathcal{A}} \Psi^*(\mathcal{C}) \rightarrow \Psi^*(\mathcal{C})$ satisfying the same conditions, so $\Psi^*(\mathcal{C})$ is thus equipped with an \mathcal{A} -algebra structure. In particular, it follows immediately from the definitions that the \mathcal{A} -algebra $\Psi^*(\mathcal{B})$ is *equal to \mathcal{A}* (up to a canonical isomorphism).

Similarly, if \mathcal{M} is a \mathcal{C} -module, the data of this module structure is the same as that of a \mathcal{B} -homomorphism $\mathcal{C} \otimes_{\mathcal{B}} \mathcal{M} \rightarrow \mathcal{M}$ satisfying the associativity condition; hence we give a $\Psi^*(\mathcal{C})$ -module structure on $\Psi^*(\mathcal{M})$.

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(4.3.5). Let \mathcal{I} be a sheaf of ideals of \mathcal{B} ; as the functor ψ^* is exact, the $\psi^*(\mathcal{B})$ -module $\psi^*(\mathcal{I})$ canonically identifies with a sheaf of ideals of $\psi^*(\mathcal{B})$; the canonical injection $\psi^*(\mathcal{I}) \rightarrow \psi^*(\mathcal{B})$ then gives a homomorphism of \mathcal{A} -modules $\Psi^*(\mathcal{I}) = \psi^*(\mathcal{I}) \otimes_{\psi^*(\mathcal{B})} \mathcal{A}_{[\theta]} \rightarrow \mathcal{A}$; we denote by $\Psi^*(\mathcal{I})\mathcal{A}$, or $\mathcal{I}\mathcal{A}$ if there is no fear of confusion, the image of $\Psi^*(\mathcal{I})$ under this homomorphism. So we have by definition $\mathcal{I}\mathcal{A} = \theta^\sharp(\psi^*(\mathcal{I}))\mathcal{A}$ and in particular, for each $x \in X$, $(\mathcal{I}\mathcal{A})_x = \theta_x^\sharp(\mathcal{I}_{\psi(x)})\mathcal{A}_x$, taking into account the canonical identification between the stalks of $\psi^*(\mathcal{I})$ and those of \mathcal{I} (3.7.2). If $\mathcal{I}_1, \mathcal{I}_2$ are two sheaves of ideals of \mathcal{B} , then we have $(\mathcal{I}_1\mathcal{I}_2)\mathcal{A} = \mathcal{I}_1(\mathcal{I}_2\mathcal{A}) = (\mathcal{I}_1\mathcal{A})(\mathcal{I}_2\mathcal{A})$.

If \mathcal{F} is an \mathcal{A} -module, we set $\mathcal{I}\mathcal{F} = (\mathcal{I}\mathcal{A})\mathcal{F}$.

(4.3.6). Let (Z, \mathcal{C}) be a third ringed space, $\Psi' = (\psi', \theta')$ a morphism $(Y, \mathcal{B}) \rightarrow (Z, \mathcal{C})$; if Ψ'' is the composite morphism $\Psi' \circ \Psi$, it follows from the definition (4.3.1) and from (4.3.3.1) that we have $\Psi''^* = \Psi^* \circ \Psi'^*$.

4.4. Relation between direct and inverse images

(4.4.1). The hypotheses and notation being the same as in (4.2.1), let \mathcal{G} be a \mathcal{B} -module. By definition, a homomorphism $u : \mathcal{G} \rightarrow \Psi_*(\mathcal{F})$ of \mathcal{B} -modules is still called a Ψ -morphisms from \mathcal{G} to \mathcal{F} , or simply a *homomorphism from \mathcal{G} to \mathcal{F}* and we write it as $u : \mathcal{G} \rightarrow \mathcal{F}$ when no confusion will occur. To give such a homomorphism is the same as giving, for each pair (U, V) where U is an open set of X , V an open set of Y such that $\psi(U) \subset V$, a *homomorphism* $u_{U,V} : \Gamma(V, \mathcal{G}) \rightarrow \Gamma(U, \mathcal{F})$ of $\Gamma(V, \mathcal{B})$ -modules, $\Gamma(U, \mathcal{F})$ being considered as a $\Gamma(V, \mathcal{B})$ -module by means of the ring homomorphism $\theta_{U,V} : \Gamma(V, \mathcal{B}) \rightarrow \Gamma(U, \mathcal{A})$; the $u_{U,V}$ must in addition render commutative the diagrams (3.5.1.1). It suffices, moreover, to define u by the data of the $u_{U,V}$ when U (resp. V) varies over a basis \mathfrak{B} (resp. \mathfrak{B}') for the topology of X (resp. Y) and to check the commutativity of (3.5.1.1) for these restrictions.

(4.4.2). Under the hypotheses of (4.2.1) and (4.2.6), let \mathcal{H} be a \mathcal{C} -module, $v : \mathcal{H} \rightarrow \Psi'_*(\mathcal{G})$ a Ψ' -morphism; then $w : \mathcal{H} \xrightarrow{v} \Psi'_*(\mathcal{G}) \xrightarrow{\Psi'_*(u)} \Psi'_*(\Psi_*(\mathcal{F}))$ is a Ψ'' -morphism which we call the *composition of u and v* .

(4.4.3). We will now see that we can define a canonical *isomorphism of bifunctors* in \mathcal{F} and \mathcal{G}

$$(4.4.3.1) \quad \text{Hom}_{\mathcal{A}}(\Psi^*(\mathcal{G}), \mathcal{F}) \simeq \text{Hom}_{\mathcal{B}}(\mathcal{G}, \Psi_*(\mathcal{F}))$$

which we denote by $v \mapsto v_\theta^\flat$ (or simply $v \mapsto v^\flat$ if there is no chance of confusion); we denote by $u \mapsto u_\theta^\sharp$, or $u \mapsto u^\sharp$, the inverse isomorphism. This definition is the following: by composing $v : \Psi^*(\mathcal{G}) \rightarrow \mathcal{F}$ with the canonical map $\psi^*(\mathcal{G}) \rightarrow \Psi^*(\mathcal{G})$, we obtain a homomorphism of sheaves of groups $v' : \psi^*(\mathcal{G}) \rightarrow \mathcal{F}$, which is also a homomorphism of $\psi^*(\mathcal{B})$ -modules. We obtain (3.7.1) a homomorphism $v'^\flat : \mathcal{G} \rightarrow \psi_*(\mathcal{F}) = \Psi_*(\mathcal{F})$, which is also a homomorphism of \mathcal{B} -modules as we check easily; we take $v_\theta^\flat = v'^\flat$. Similarly, for $u : \mathcal{G} \rightarrow \Psi_*(\mathcal{F})$, which is a homomorphism of \mathcal{B} -modules, we obtain (3.7.1) a homomorphism $u^\sharp : \psi^*(\mathcal{G}) \rightarrow \mathcal{F}$ of $\psi^*(\mathcal{B})$ -modules, hence by tensoring with \mathcal{A} we have a homomorphism of \mathcal{A} -modules $\Psi^*(\mathcal{G}) \rightarrow \mathcal{F}$, which we denote by u_θ^\sharp . It is immediate to check that $(u_\theta^\sharp)_\theta^\flat = u$ and $(v_\theta^\flat)_\theta^\sharp = v$, so we have established the functorial nature in \mathcal{F} of the isomorphism $v \mapsto v_\theta^\flat$. The functorial nature in \mathcal{G} of $u \mapsto u_\theta^\sharp$ is then formally shown as in (3.5.4) (reasoning that would also prove the functorial nature of Ψ^* established in (4.3.1) directly).

If we take for v the identity homomorphism of $\Psi^*(\mathcal{B})$, v_θ^\flat is a homomorphism

$$(4.4.3.2) \quad \rho_{\mathcal{G}} : \mathcal{G} \longrightarrow \Psi_*(\Psi^*(\mathcal{G}));$$

if we take for u the identity homomorphism of $\Psi_*(\mathcal{F})$, u_θ^\sharp is a homomorphism

$$(4.4.3.3) \quad \sigma_{\mathcal{F}} : \Psi^*(\Psi_*(\mathcal{F})) \longrightarrow \mathcal{F};$$

these homomorphisms will be called *canonical*. They are in general neither injective or surjective. We have canonical factorizations analogous to (3.5.3.3) and (3.5.4.4).

We note that if s is a section of \mathcal{G} over an open set V of Y , $\rho_{\mathcal{G}}(s)$ is the section $s' \otimes 1$ of $\Psi^*(\mathcal{G})$ over $\psi^{-1}(V)$, s' being such that $s'_x = s_{\psi(x)}$ for all $x \in \psi^{-1}(V)$. We also note that if $u : \mathcal{G} \rightarrow \Psi_*(\mathcal{F})$ is a homomorphism, it defines for all $x \in X$ a homomorphism $u_x : \mathcal{G}_{\psi(x)} \rightarrow \mathcal{F}_x$ on the stalks, obtained by composing $(u^\sharp)_x : (\Psi^*(\mathcal{G}))_x \rightarrow \mathcal{F}_x$ and the canonical homomorphism $s_x \mapsto s_x \otimes 1$ from $\mathcal{G}_{\psi(x)}$ to

$(\Psi^*(\mathcal{G}))_x = \mathcal{G}_{\psi(x)} \otimes_{\mathcal{B}_{\psi(x)}} \mathcal{A}_x$. The homomorphism u_x is obtained also by passing to the inductive limit relative to the homomorphisms $\Gamma(V, \mathcal{G}) \xrightarrow{u} \Gamma(\psi^{-1}(V), \mathcal{F}) \rightarrow \mathcal{F}_x$, where V varies over the neighbourhoods of $\psi(x)$.

(4.4.4). Let $\mathcal{F}_1, \mathcal{F}_2$ be \mathcal{A} -modules, $\mathcal{G}_1, \mathcal{G}_2$ be \mathcal{B} -modules, u_i ($i = 1, 2$) a homomorphism from \mathcal{G}_i to \mathcal{F}_i . We denote by $u_1 \otimes u_2$ the homomorphism $u : \mathcal{G}_1 \otimes_{\mathcal{B}} \mathcal{G}_2 \rightarrow \mathcal{F}_1 \otimes_{\mathcal{A}} \mathcal{F}_2$ such that $u^\sharp = (u_1)^\sharp \otimes (u_2)^\sharp$ (taking into account (4.3.3.1)); we check that u is also the composition $\mathcal{G}_1 \otimes_{\mathcal{B}} \mathcal{G}_2 \rightarrow \Psi_*(\mathcal{F}_1) \otimes_{\mathcal{B}} \Psi_*(\mathcal{F}_2) \rightarrow \Psi_*(\mathcal{F}_1 \otimes_{\mathcal{A}} \mathcal{F}_2)$, where the first arrow is the ordinary tensor product $u_1 \otimes_{\mathcal{B}} u_2$ and the second is the canonical homomorphism (4.2.2.1).

(4.4.5). Let $(\mathcal{G}_\lambda)_{\lambda \in L}$ be an inductive system of \mathcal{B} -modules, and, for each $\lambda \in L$, let u_λ be a homomorphism $\mathcal{G}_\lambda \rightarrow \Psi_*(\mathcal{F})$, form an inductive limit; we put $\mathcal{G} = \varinjlim \mathcal{G}_\lambda$ and $u = \varinjlim u_\lambda$; then the $(u_\lambda)^\sharp$ form an inductive system of homomorphisms $\Psi^*(\mathcal{G}_\lambda) \rightarrow \mathcal{F}$, and the inductive limit of this system is none other than u^\sharp .

(4.4.6). Let \mathcal{M}, \mathcal{N} be two \mathcal{B} -modules, V an open set of Y , $U = \psi^{-1}(V)$; the map $v \mapsto \Psi^*(v)$ is a homomorphism

$$\mathrm{Hom}_{\mathcal{B}|V}(\mathcal{M}|V, \mathcal{N}|V) \longrightarrow \mathrm{Hom}_{\mathcal{A}|U}(\Psi^*(\mathcal{M})|U, \Psi^*(\mathcal{N})|U)$$

for the $\Gamma(V, \mathcal{B})$ -module structures ($\mathrm{Hom}_{\mathcal{A}|U}(\Psi^*(\mathcal{M})|U, \Psi^*(\mathcal{N})|U)$ is normally equipped with the a $\Gamma(U, \psi^*(\mathcal{B}))$ -module structure, and thanks to the canonical homomorphism (3.7.2) $\Gamma(V, \mathcal{B}) \rightarrow \Gamma(U, \psi^*(\mathcal{B}))$, it is also a $\Gamma(V, \mathcal{B})$ -module). We see immediately that these homomorphisms are compatible with the restriction morphisms, and as a result define a canonical functorial homomorphism

$$\gamma : \mathrm{Hom}_{\mathcal{B}}(\mathcal{M}, \mathcal{N}) \longrightarrow \Psi_*(\mathrm{Hom}_{\mathcal{A}}(\Psi^*(\mathcal{M}), \Psi^*(\mathcal{N})));$$

it also corresponds to this homomorphism the homomorphism

$$\gamma^\sharp : \Psi^*(\mathrm{Hom}_{\mathcal{B}}(\mathcal{M}, \mathcal{N})) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(\Psi^*(\mathcal{M}), \Psi^*(\mathcal{N}))$$

and these canonical morphisms are functorial in \mathcal{M} and \mathcal{N} .

(4.4.7). Suppose that \mathcal{F} (resp. \mathcal{G}) is an \mathcal{A} -algebra (resp. a \mathcal{B} -algebra). If $u : \mathcal{G} \rightarrow \Psi_*(\mathcal{F})$ is a homomorphism of \mathcal{B} -algebras, u^\sharp is a homomorphism $\Psi^*(\mathcal{G}) \rightarrow \mathcal{F}$ of \mathcal{A} -algebras; this follows from the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{G} \otimes_{\mathcal{B}} \mathcal{G} & \longrightarrow & \mathcal{G} \\ \downarrow & & \downarrow u \\ \Psi_*(\mathcal{F} \otimes_{\mathcal{A}} \mathcal{F}) & \longrightarrow & \Psi_*(\mathcal{F}) \end{array}$$

and from (4.4.4). Similarly, if $v : \Psi^*(\mathcal{G}) \rightarrow \mathcal{F}$ is a homomorphism of \mathcal{A} -algebras, $v^\flat : \mathcal{G} \rightarrow \Psi_*(\mathcal{F})$ is a homomorphism of \mathcal{B} -algebras.

(4.4.8). Let (Z, \mathcal{C}) be a third ringed space, $\Psi' = (\psi', \theta')$ a morphism $(Y, \mathcal{B}) \rightarrow (Z, \mathcal{C})$, and $\Psi'' : (X, \mathcal{A}) \rightarrow (Z, \mathcal{C})$ the composite morphism $\Psi' \circ \Psi$. Let \mathcal{H} be a \mathcal{C} -module, u' a homomorphism from \mathcal{H} to \mathcal{G} ; the composition $v'' = v \circ v'$ is by definition the homomorphism from \mathcal{H} to \mathcal{F} defined by $\mathcal{H} \xrightarrow{v'} \Psi'_*(\mathcal{G}) \xrightarrow{\Psi'_*(v)} \Psi'_*(\Psi_*(\mathcal{F}))$; we check that v''^\sharp is the homomorphism

$$\Psi^*(\Psi'^*(\mathcal{H})) \xrightarrow{\Psi^*(v'^\sharp)} \Psi^*(\mathcal{G}) \xrightarrow{v^\sharp} \mathcal{F}.$$

§5. QUASI-COHERENT AND COHERENT SHEAVES

5.1. Quasi-coherent sheaves

(5.1.1). Let (X, \mathcal{O}_X) be a ringed space, \mathcal{F} an \mathcal{O}_X -module. The data of a homomorphism $u : \mathcal{O}_X \rightarrow \mathcal{F}$ of \mathcal{O}_X -modules is equivalent to that of the section $s = u(1) \in \Gamma(X, \mathcal{F})$. Indeed, when s is given, for each section $t \in \Gamma(U, \mathcal{O}_X)$, we necessarily have $u(t) = t \cdot (s|U)$; we say that u is *defined by the section* s . If now I is any set of indices, consider the direct sum sheaf $\mathcal{O}_X^{(I)}$, and for each $i \in I$, let h_i be the canonical injection of the i -th factor into $\mathcal{O}_X^{(I)}$; we know that $u \mapsto (u \circ h_i)$ is an

isomorphism from $\text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X^{(I)}, \mathcal{F})$ to the product $(\text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}))^I$. So there is a canonical one-to-one correspondence between the homomorphisms $u : \mathcal{O}_X^{(I)} \rightarrow \mathcal{F}$ and the families of sections $(s_i)_{i \in I}$ of \mathcal{F} over X . The homomorphism u corresponding to (s_i) sends an element $(a_i) \in (\Gamma(U, \mathcal{O}_X))^{(I)}$ to $\sum_{i \in I} a_i \cdot (s_i|_U)$.

We say that \mathcal{F} is *generated by the family* (s_i) if the homomorphism $\mathcal{O}_X^{(I)} \rightarrow \mathcal{F}$ defined for each family is *surjective* (in other words, if, for each $x \in X$, \mathcal{F}_x is an \mathcal{O}_x -module generated by the $(s_i)_x$). We say that \mathcal{F} is *generated by its sections over* X if it is generated by the family of all these sections (or by a subfamily), in other words, if there exists a surjective homomorphism $\mathcal{O}_X^{(I)} \rightarrow \mathcal{F}$ for a suitable I .

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We note that an \mathcal{O}_X -module \mathcal{F} can be such that there exists a point $x_0 \in X$ for which $\mathcal{F}|_U$ is not generated by its sections over U , *regardless of the choice of neighbourhood* U of x_0 : it suffices to take $X = \mathbf{R}$, for \mathcal{O}_X the simple sheaf \mathbf{Z} , for \mathcal{F} the algebraic subsheaf of \mathcal{O}_X such that $\mathcal{F}_0 = \{0\}$, $\mathcal{F}_x = \mathbf{Z}$ for $x \neq 0$, and finally $x_0 = 0$: the only section of $\mathcal{F}|_U$ over U is 0 for a neighbourhood U of 0.

(5.1.2). Let $f : X \rightarrow Y$ be a morphism of ringed spaces. If \mathcal{F} is an \mathcal{O}_X -module generated by its sections over X , then the canonical homomorphism $f^*(f_*(\mathcal{F})) \rightarrow \mathcal{F}$ (4.4.3.3) is *surjective*; indeed, with the notation of (5.1.1), $s_i \otimes 1$ is a section of $f^*(f_*(\mathcal{F}))$ over X , and its image in \mathcal{F} is s_i . The example in (5.1.1) where f is the identity shows that the inverse of this proposition is false in general.

If \mathcal{G} is an \mathcal{O}_Y -module generated by its sections over Y , then $f^*(\mathcal{G})$ is generated by its sections over X , since f^* is a right exact functor.

ErrII

(5.1.3). We say that an \mathcal{O}_X -module \mathcal{F} is *quasi-coherent* if for each $x \in X$ there is an open neighbourhood U of x such that $\mathcal{F}|_U$ is isomorphic to the *cokernel* of a homomorphism of the form $\mathcal{O}_X^{(I)}|_U \rightarrow \mathcal{O}_X^{(J)}|_U$, where I and J are sets of arbitrary indices. It is clear that \mathcal{O}_X is itself a quasi-coherent \mathcal{O}_X -module, and that any direct sum of quasi-coherent \mathcal{O}_X -modules is again a quasi-coherent \mathcal{O}_X -module. We say that an \mathcal{O}_X -algebra \mathcal{A} is *quasi-coherent* if it is quasi-coherent as an \mathcal{O}_X -module.

(5.1.4). Let $f : X \rightarrow Y$ be a morphism of ringed spaces. If \mathcal{G} is a quasi-coherent \mathcal{O}_Y -module, then $f^*(\mathcal{G})$ is a quasi-coherent \mathcal{O}_X -module. Indeed, for each $x \in X$, there is an open neighbourhood V of $f(x)$ in Y such that $\mathcal{G}|_V$ is the cokernel of a homomorphism $\mathcal{O}_Y^{(I)}|_V \rightarrow \mathcal{O}_Y^{(J)}|_V$. If $U = f^{-1}(V)$, and if f_U is the restriction of f to U , then we have $f^*(\mathcal{G})|_U = f_U^*(\mathcal{G}|_V)$; as f_U^* is right exact and commutes with direct sums, $f_U^*(\mathcal{G}|_V)$ is the cokernel of a homomorphism $\mathcal{O}_X^{(I)}|_U \rightarrow \mathcal{O}_X^{(J)}|_U$.

5.2. Sheaves of finite type

(5.2.1). We say that an \mathcal{O}_X -module \mathcal{F} is of *finite type* if for each $x \in X$ there exists an open neighbourhood U of x such that $\mathcal{F}|_U$ is generated by a *finite* family of sections over U , or if it is isomorphic to a sheaf quotient of a sheaf of the form $(\mathcal{O}_X|_U)^p$ where p is finite. Each sheaf quotient of a sheaf of finite type is again a sheaf of finite type, as well as each finite direct sum and each finite tensor product of sheaves of finite type. An \mathcal{O}_X -module of finite type is not necessarily quasi-coherent, as we can see for the \mathcal{O}_X -module $\mathcal{O}_X/\mathcal{F}$, where \mathcal{F} is the example in (5.1.1). If \mathcal{F} is of finite type, then \mathcal{F}_x is an \mathcal{O}_x -module of finite type for each $x \in X$, but the example in (5.1.1) shows that this condition is necessary but not sufficient in general.

(5.2.2). Let \mathcal{F} be an \mathcal{O}_X -module of *finite type*. If s_i ($1 \leq i \leq n$) are the sections of \mathcal{F} over an open neighbourhood U of a point $x \in X$ and the $(s_i)_x$ generate \mathcal{F}_x , then there exists an open neighbourhood $V \subset U$ of x such that the $(s_i)_y$ generate \mathcal{F}_y for all $y \in V$ (FAC, I, 2, 12, prop. 1). In particular, we conclude that the support of \mathcal{F} is *closed*.

Similarly, if $u : \mathcal{F} \rightarrow \mathcal{G}$ is a homomorphism such that $u_x = 0$, then there exists a neighbourhood U of x such that $u_y = 0$ for all $y \in U$.

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(5.2.3). Suppose that X is *quasi-compact*, and let \mathcal{F} and \mathcal{G} be two \mathcal{O}_X -modules such that \mathcal{G} is of *finite type*, $u : \mathcal{F} \rightarrow \mathcal{G}$ a *surjective* homomorphism. In addition, suppose that \mathcal{F} is the inductive limit of an inductive system (\mathcal{F}_λ) of \mathcal{O}_X -modules. Then there exists an index μ such that the homomorphism $\mathcal{F}_\mu \rightarrow \mathcal{G}$ is *surjective*. Indeed, for each $x \in X$, there exists a finite system of sections s_i of \mathcal{G} over an open neighbourhood $U(x)$ of x such that the $(s_i)_y$ generate \mathcal{G}_y for all $y \in U(x)$; there is then an open

neighbourhood $V(x) \subset U(x)$ of x and n sections t_i of \mathcal{F} over $V(x)$ such that $s_i|V(s) = u(t_i)$ for all i ; we can also suppose that the t_i are the canonical images of sections of a similar sheaf $\mathcal{F}_{\lambda(x)}$ over $V(x)$. We then cover X with a finite number of neighbourhoods $V(x_k)$, and let μ be the maximal index of the $\lambda(x_k)$; it is clear that this index gives the answer.

Suppose still that X is quasi-compact, and let \mathcal{F} be an \mathcal{O}_X -module of finite type generated by its sections over X (5.1.1); then \mathcal{F} is generated by a *finite* subfamily of these sections: indeed, it suffices to cover X by a finite number of open neighbourhoods U_k such that, for each k , there is a finite number of sections s_{ik} of \mathcal{F} over X whose restrictions to U_k generate $\mathcal{F}|U_k$; it is clear that the s_{ik} then generate \mathcal{F} .

(5.2.4). Let $f : X \rightarrow Y$ be a morphism of ringed spaces. If \mathcal{G} is an \mathcal{O}_Y -module of finite type, then $f^*(\mathcal{G})$ is an \mathcal{O}_X -module of finite type. Indeed, for each $x \in X$, there is an open neighbourhood V of $f(x)$ in Y and a surjective homomorphism $v : \mathcal{O}_Y^p|V \rightarrow \mathcal{G}|V$. If $U = f^{-1}(V)$ and if f_U is the restriction of f to U , then we have $f^*(\mathcal{G})|U = f_U^*(\mathcal{G}|V)$; since f_U^* is right exact (4.3.1) and commutes with direct sums (4.3.2), $f_U^*(v)$ is a surjective homomorphism $\mathcal{O}_X^p|U \rightarrow f^*(\mathcal{G})|U$.

ErrII

(5.2.5). We say that an \mathcal{O}_X -module \mathcal{F} admits a *finite presentation* if for each $x \in X$ there exists an open neighbourhood U of x such that $\mathcal{F}|U$ is isomorphic to a *cokernel* of a $(\mathcal{O}_X|U)$ -homomorphism $\mathcal{O}_X^p|U \rightarrow \mathcal{O}_X^q|U$, p and q being two integers > 0 . Such an \mathcal{O}_X -module is therefore of finite type and quasi-coherent. If $f : X \rightarrow Y$ is a morphism of ringed spaces, and if \mathcal{G} is an \mathcal{O}_Y -module admitting a finite presentation, then $f^*(\mathcal{G})$ admits a finite presentation, as shown in the argument of (5.1.4).

(5.2.6). Let \mathcal{F} be an \mathcal{O}_X -module admitting a finite presentation (5.2.5); then, for each \mathcal{O}_X -module \mathcal{H} , the canonical functorial homomorphism

$$(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{H}))_x \longrightarrow \mathcal{H}om_{\mathcal{O}_x}(\mathcal{F}_x, \mathcal{H}_x)$$

is *bijective* (T, 4.1.1).

(5.2.7). Let \mathcal{F} and \mathcal{G} be two \mathcal{O}_X -modules admitting a finite presentation. If for some $x \in X$, \mathcal{F}_x and \mathcal{G}_x are *isomorphic* as \mathcal{O}_x -modules, then there exists an open neighbourhood U of x such that $\mathcal{F}|U$ and $\mathcal{G}|U$ are *isomorphic*. Indeed, if $\phi : \mathcal{F}_x \rightarrow \mathcal{G}_x$ and $\psi : \mathcal{G}_x \rightarrow \mathcal{F}_x$ are an isomorphism and its inverse isomorphism, then there exists, according to (5.2.6), an open neighbourhood V of x and a section u (resp. v) of $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ (resp. $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F})$) over V such that $u_x = \phi$ (resp. $v_x = \psi$). As $(u \circ v)_x$ and $(v \circ u)_x$ are the identity automorphisms, there exists an open neighbourhood $U \subset V$ of x such that $(u \circ v)|U$ and $(v \circ u)|U$ are the identity automorphisms, hence the proposition.

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5.3. Coherent sheaves

(5.3.1). We say that an \mathcal{O}_X -module \mathcal{F} is *coherent* if it satisfies the two following conditions:

- (a) \mathcal{F} is of finite type.
- (b) for each open $U \subset X$, integer $n > 0$, and homomorphism $u : \mathcal{O}_X^n|U \rightarrow \mathcal{F}|U$, the kernel of u is of finite type.

We note that these two conditions are of a *local* nature.

For most of the proofs of the properties of coherent sheaves in what follows, cf. (FAC, I, 2).

(5.3.2). Each coherent \mathcal{O}_X -module admits a finite presentation (5.2.5); the inverse is not necessarily true, since \mathcal{O}_X itself is not necessarily a coherent \mathcal{O}_X -module.

Each \mathcal{O}_X -submodule of *finite type* of a coherent \mathcal{O}_X -module is coherent; each *finite* direct sum of coherent \mathcal{O}_X -modules is a coherent \mathcal{O}_X -module.

(5.3.3). If $0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow 0$ is an exact sequence of \mathcal{O}_X -modules and if two of these \mathcal{O}_X -modules are coherent, then so is the third.

(5.3.4). If \mathcal{F} and \mathcal{G} are two coherent \mathcal{O}_X -modules, $u : \mathcal{F} \rightarrow \mathcal{G}$ a homomorphism, then $\text{Im}(u)$, $\text{Ker}(u)$, and $\text{Coker}(u)$ are coherent \mathcal{O}_X -modules. In particular, if \mathcal{F} and \mathcal{G} are \mathcal{O}_X -submodules of a coherent \mathcal{O}_X -module, then $\mathcal{F} + \mathcal{G}$ and $\mathcal{F} \cap \mathcal{G}$ are coherent.

If $\mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C} \rightarrow \mathcal{D} \rightarrow \mathcal{E}$ is an exact sequence of \mathcal{O}_X -modules, and if \mathcal{A} , \mathcal{B} , \mathcal{D} , \mathcal{E} are coherent, then \mathcal{C} is coherent.

ErrII

(5.3.5). If \mathcal{F} and \mathcal{G} are two coherent \mathcal{O}_X -modules, then so are $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$.

(5.3.6). Let \mathcal{F} be a coherent \mathcal{O}_X -module, \mathcal{J} a coherent sheaf of ideals of \mathcal{O}_X . Then the \mathcal{O}_X -module $\mathcal{J}\mathcal{F}$ is coherent, as the image of $\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{F}$ under the canonical homomorphism $\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{F} \rightarrow \mathcal{F}$ ((5.3.4) and (5.3.5)).

(5.3.7). We say that an \mathcal{O}_X -algebra \mathcal{A} is *coherent* if it is coherent as an \mathcal{O}_X -module. In particular, \mathcal{O}_X is a *coherent sheaf of rings* if and only if for each open $U \subset X$ and each homomorphism of the form $u : \mathcal{O}_X^p|U \rightarrow \mathcal{O}_X|U$, the kernel of u is an $(\mathcal{O}_X|U)$ -module of finite type.

If \mathcal{O}_X is a coherent sheaf of rings, then each \mathcal{O}_X -module \mathcal{F} admitting a finite presentation (5.2.5) is coherent, according to (5.3.4).

The *annihilator* of an \mathcal{O}_X -module \mathcal{F} is the kernel \mathcal{J} of the canonical homomorphism $\mathcal{O}_X \rightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$ which sends each section $s \in \Gamma(U, \mathcal{O}_X)$ to the multiplication by s map in $\mathcal{H}om(\mathcal{F}|U, \mathcal{F}|U)$; if \mathcal{O}_X is coherent and if \mathcal{F} is a coherent \mathcal{O}_X -module, then \mathcal{J} is coherent ((5.3.4) and (5.3.5)) and for each $x \in X$, \mathcal{J}_x is the annihilator of \mathcal{F}_x (5.2.6).

(5.3.8). Suppose that \mathcal{O}_X is coherent; let \mathcal{F} be a coherent \mathcal{O}_X -module, x a point of X , M a submodule of finite type of \mathcal{F}_x ; then there exists an open neighbourhood U of x and a coherent $(\mathcal{O}_X|U)$ -submodule \mathcal{G} of $\mathcal{F}|U$ such that $\mathcal{G}_x = M$ (T, 4.1, Lemma 1).

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This result, along with the properties of the \mathcal{O}_X -submodules of a coherent \mathcal{O}_X -module, impose the necessary conditions on the rings \mathcal{O}_x such that \mathcal{O}_X is coherent. For example (5.3.4), the intersection of two ideals of finite type of \mathcal{O}_x must still be an ideal of finite type.

(5.3.9). Suppose that \mathcal{O}_X is coherent, and let M be an \mathcal{O}_x -module admitting a finite presentation, therefore isomorphic to a cokernel of a homomorphism $\phi : \mathcal{O}_x^p \rightarrow \mathcal{O}_x^q$; then there exists an open neighbourhood U of x and a coherent $(\mathcal{O}_X|U)$ -module \mathcal{F} such that \mathcal{F}_x is isomorphic to M . Indeed, according to (5.2.6), there exists a section u of $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X^p, \mathcal{O}_X^q)$ over an open neighbourhood U of x such that $u_x = \phi$; the cokernel \mathcal{F} of the homomorphism $u : \mathcal{O}_X^p|U \rightarrow \mathcal{O}_X^q|U$ gives the answer (5.3.4).

(5.3.10). Suppose that \mathcal{O}_X is coherent, and let \mathcal{J} be a coherent sheaf of ideals of \mathcal{O}_X . For a $(\mathcal{O}_X/\mathcal{J})$ -module \mathcal{F} to be coherent, it is necessary and sufficient for it to be coherent as an \mathcal{O}_X -module. In particular, $\mathcal{O}_X/\mathcal{J}$ is a coherent sheaf of rings.

(5.3.11). Let $f : X \rightarrow Y$ be a morphism of ringed spaces, and suppose that \mathcal{O}_X is coherent; then, for each coherent \mathcal{O}_Y -module \mathcal{G} , $f^*(\mathcal{G})$ is a coherent \mathcal{O}_X -module. Indeed, with the notation of (5.2.4), we can assume that $\mathcal{G}|V$ is the cokernel of a homomorphism $v : \mathcal{O}_Y^q|V \rightarrow \mathcal{O}_Y^p|V$; as f_U^* is right exact, $f^*(\mathcal{G})|U = f_U^*(\mathcal{G}|V)$ is the cokernel of the homomorphism $f_U^*(v) : \mathcal{O}_X^q|U \rightarrow \mathcal{O}_X^p|U$, hence our assertion.

(5.3.12). Let Y be a closed subset of X , $j : Y \rightarrow X$ the canonical injection, \mathcal{O}_Y a sheaf of rings on Y , and set $\mathcal{O}_X = j_*(\mathcal{O}_Y)$. For an \mathcal{O}_Y -module \mathcal{G} to be of finite type (resp. quasi-coherent, coherent), it is necessary and sufficient for $j_*(\mathcal{G})$ to be an \mathcal{O}_X -module of finite type (resp. quasi-coherent, coherent).

5.4. Locally free sheaves

(5.4.1). Let X be a ringed space. We say that an \mathcal{O}_X -module \mathcal{F} is *locally free* if for each $x \in X$ there exists an open neighbourhood U of x such that $\mathcal{F}|U$ is isomorphic to a $(\mathcal{O}_X|U)$ -module of the form $\mathcal{O}_X^{(I)}|U$, where I can depend on U . If for each U , I is finite, then we say that \mathcal{F} is *of finite rank*; if for each U , I has the same finite number of elements n , we say that \mathcal{F} is *of rank n* . A locally free \mathcal{O}_X -module of rank 1 is called *invertible* (cf. (5.4.3)). If \mathcal{F} is a locally free \mathcal{O}_X -module of finite rank, then for each $x \in X$, \mathcal{F}_x is a free \mathcal{O}_x -module of finite rank $n(x)$, and there exists a neighbourhood U of x such that $\mathcal{F}|U$ is of rank $n(x)$; if X is connected, then $n(x)$ is *constant*.

It is clear that each locally free sheaf is quasi-coherent, and if \mathcal{O}_X is a coherent sheaf of rings, then each locally free \mathcal{O}_X -module of finite rank is coherent.

If \mathcal{L} is locally free, then $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{F}$ is an *exact* functor in \mathcal{F} to the category of \mathcal{O}_X -modules.

We will mostly consider locally free \mathcal{O}_X -modules of finite rank, and when we speak of locally free sheaves without specifying, it will be understood that they are of *finite rank*.

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Suppose that \mathcal{O}_X is *coherent*, and let \mathcal{F} be a *coherent* \mathcal{O}_X -module. Then, if at a point $x \in X$, \mathcal{F}_x is an \mathcal{O}_x -module *free of rank n* , there exists a neighbourhood U of x such that $\mathcal{F}|U$ is *locally free of rank n* ; in fact, \mathcal{F}_x is then isomorphic to \mathcal{O}_x^n , and the proposition follows from (5.2.7).

ErrII

(5.4.2). If \mathcal{L}, \mathcal{F} are two \mathcal{O}_X -modules, we have a canonical functorial homomorphism

$$(5.4.2.1) \quad \mathcal{L}^\vee \otimes_{\mathcal{O}_X} \mathcal{F} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{F})$$

defined in the following way: for each open set U , send any pair (u, t) , where $u \in \Gamma(U, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)) = \text{Hom}(\mathcal{L}|_U, \mathcal{O}_X|_U)$ and $t \in \Gamma(U, \mathcal{F})$, to the element of $\text{Hom}(\mathcal{L}|_U, \mathcal{F}|_U)$ which, for each $x \in U$, sends $s_x \in \mathcal{L}_x$ to the element $u_x(s_x)t_x$ of \mathcal{F}_x . If \mathcal{L} is *locally free of finite rank*, then this homomorphism is *bijective*; the property being local, we can in fact reduce to the case where $\mathcal{L} = \mathcal{O}_X^n$; as for each \mathcal{O}_X -module \mathcal{G} , $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X^n, \mathcal{G})$ is canonically isomorphic to \mathcal{G}^n , we have reduced to the case $\mathcal{L} = \mathcal{O}_X$, which is immediate.

(5.4.3). If \mathcal{L} is invertible, then so is its dual $\mathcal{L}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$, since we can immediately reduce (as the question is local) to the case $\mathcal{L} = \mathcal{O}_X$. In addition, we have a canonical isomorphism

$$(5.4.3.1) \quad \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{L} \simeq \mathcal{O}_X$$

as, according to (5.3.2), it suffices to define a canonical isomorphism $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L}) \simeq \mathcal{O}_X$. For each \mathcal{O}_X -module \mathcal{F} , we have a canonical homomorphism $\mathcal{O}_X \simeq \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$ (5.3.7). It remains to prove that if $\mathcal{F} = \mathcal{L}$ is invertible, then this homomorphism is bijective, and as the question is local, it reduces to the case $\mathcal{L} = \mathcal{O}_X$, which is immediate.

Due to the above, we put $\mathcal{L}^{-1} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$, and we say that \mathcal{L}^{-1} is the *inverse* of \mathcal{L} . The terminology “invertible sheaf” can be justified in the following way when X is a point and \mathcal{O}_X is a *local ring* A with maximal ideal \mathfrak{m} ; if M and M' are two A -modules (M being of finite type) such that $M \otimes_A M'$ is isomorphic to A , then as $(A/\mathfrak{m}) \otimes_A (M \otimes_A M')$ identifies with $(M/\mathfrak{m}M) \otimes_{A/\mathfrak{m}} (M'/\mathfrak{m}M')$, this latter tensor product of vector spaces over the field A/\mathfrak{m} is isomorphic to A/\mathfrak{m} , which requires $M/\mathfrak{m}M$ and $M'/\mathfrak{m}M'$ to be of dimension 1. For each element $z \in M$ not in $\mathfrak{m}M$, we have $M = Az + \mathfrak{m}M$, which implies that $M = Az$ according to Nakayama’s Lemma, M being of finite type. Moreover, as the annihilator of z kills $M \otimes_A M'$, which is isomorphic to A , this annihilator is $\{0\}$, and as a result M is *isomorphic* to A . In the general case, this shows that \mathcal{L} is an \mathcal{O}_X -module of finite type, such that there exists an \mathcal{O}_X -module \mathcal{F} for which $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{F}$ is isomorphic to \mathcal{O}_X , and if in addition the rings \mathcal{O}_x are local rings, then \mathcal{L}_x is an \mathcal{O}_x -module isomorphic to \mathcal{O}_x for each $x \in X$. If \mathcal{O}_X and \mathcal{L} are assumed to be *coherent*, then we conclude that \mathcal{L} is invertible according to (5.2.7).

(5.4.4). If \mathcal{L} and \mathcal{L}' are two invertible \mathcal{O}_X -modules, then so is $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}'$, since the question is local, we can assume that $\mathcal{L} = \mathcal{O}_X$, and the result is then trivial. For each integer $n \geq 1$, we denote by $\mathcal{L}^{\otimes n}$ the tensor product of n copies of the sheaf \mathcal{L} ; we set by convention $\mathcal{L}^{\otimes 0} = \mathcal{O}_X$, and for $n \geq 1$, $\mathcal{L}^{\otimes(-n)} = (\mathcal{L}^{-1})^{\otimes n}$. With these notation, there is then a *canonical functorial isomorphism*

$$(5.4.4.1) \quad \mathcal{L}^{\otimes m} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n} \simeq \mathcal{L}^{\otimes(n+m)}$$

for any rational integers m and n : indeed, by definition, we immediately reduce to the case where $m = -1, n = 1$, and the isomorphism in question is then that defined in (5.4.3).

(5.4.5). Let $f : Y \rightarrow X$ be a morphism of ringed spaces. If \mathcal{L} is a locally free (resp. invertible) \mathcal{O}_X -module, then $f^*(\mathcal{L})$ is a locally free (resp. invertible) \mathcal{O}_Y -module: this follows immediately from that the inverse images of two locally isomorphic \mathcal{O}_X -modules are locally isomorphic, that f^* commutes with finite direct sums, and that $f^*(\mathcal{O}_X) = \mathcal{O}_Y$ (4.3.4). In addition, we know that we have a canonical functorial homomorphism $f^*(\mathcal{L}^\vee) \rightarrow (f^*(\mathcal{L}))^\vee$ (4.4.6), and when \mathcal{L} is locally free, this homomorphism is *bijective*: indeed, we again reduce to the case where $\mathcal{L} = \mathcal{O}_X$ which is trivial. We conclude that if \mathcal{L} is invertible, then $f^*(\mathcal{L}^{\otimes n})$ canonically identifies with $(f^*(\mathcal{L}))^{\otimes n}$ for each rational integer n .

(5.4.6). Let \mathcal{L} be an invertible \mathcal{O}_X -module; we denote by $\Gamma_\bullet(X, \mathcal{L})$ or simply $\Gamma_\bullet(\mathcal{L})$ the abelian group direct sum $\bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{L}^{\otimes n})$; we equip it with the structure of a *graded ring*, by corresponding to a pair (s_n, s_m) , where $s_n \in \Gamma(X, \mathcal{L}^{\otimes n}), s_m \in \Gamma(X, \mathcal{L}^{\otimes m})$, the section of $\mathcal{L}^{\otimes(n+m)}$ over X which corresponds canonically (5.4.4.1) to the section $s_n \otimes s_m$ of $\mathcal{L}^{\otimes n} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes m}$; the associativity of this multiplication is verified in an immediate way. It is clear that $\Gamma_\bullet(X, \mathcal{L})$ is a covariant functor in \mathcal{L} , with values in the category of graded rings.

If now \mathcal{F} is any \mathcal{O}_X -module, then we set

$$\Gamma_\bullet(\mathcal{L}, \mathcal{F}) = \bigoplus_{n \in \mathbf{Z}} \Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}).$$

We equip this abelian group with the structure of a *graded module* over the graded ring $\Gamma_\bullet(\mathcal{L})$ in the following way: to a pair (s_n, u_m) , where $s_n \in \Gamma(X, \mathcal{L}^{\otimes n})$ and $u_m \in \Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes m})$, we associate the section of $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes(m+n)}$ which canonically corresponds (5.4.4.1) to $s_n \otimes u_m$; the verification of the module axioms are immediate. For X and \mathcal{L} fixed, $\Gamma_\bullet(\mathcal{L}, \mathcal{F})$ is a covariant functor in \mathcal{F} with values in the category of graded $\Gamma_\bullet(\mathcal{L})$ -modules; for X and \mathcal{F} fixed, it is a covariant functor in \mathcal{L} with values in the category of abelian groups.

If $f : Y \rightarrow X$ is a morphism of ringed spaces, the canonical homomorphism (4.4.3.2) $\rho : \mathcal{L}^{\otimes n} \rightarrow f_*(f^*(\mathcal{L}^{\otimes n}))$ defines a homomorphism of abelian groups $\Gamma(X, \mathcal{L}^{\otimes n}) \rightarrow \Gamma(Y, f^*(\mathcal{L}^{\otimes n}))$, and as $f^*(\mathcal{L}^{\otimes n}) = (f^*(\mathcal{L}))^{\otimes n}$, it follows from the definitions of the canonical homomorphisms (4.4.3.2) and (5.4.4.1) that the above homomorphisms define a *functorial homomorphism of graded rings* $\Gamma_\bullet(\mathcal{L}) \rightarrow \Gamma_\bullet(f^*(\mathcal{L}))$. The same canonical homomorphism (4.4.3) similarly defines a homomorphism of abelian groups $\Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) \rightarrow \Gamma(Y, f^*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}))$, and as

$$f^*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) = f^*(\mathcal{F}) \otimes_{\mathcal{O}_Y} (f^*(\mathcal{L}))^{\otimes n} \quad (4.3.3.1),$$

these homomorphism (for n variable) define a *di-homomorphism of graded modules* $\Gamma_\bullet(\mathcal{L}, \mathcal{F}) \rightarrow \Gamma_\bullet(f^*(\mathcal{L}), f^*(\mathcal{F}))$. 01 | 51

(5.4.7). One can show that there exists a *set* \mathfrak{M} (also denoted $\mathfrak{M}(X)$) of invertible \mathcal{O}_X -modules such that each invertible \mathcal{O}_X -module is isomorphic to a unique element of \mathfrak{M} ⁸; we define on \mathfrak{M} a composition law by sending two elements \mathcal{L} and \mathcal{L}' of \mathfrak{M} to the unique element of \mathfrak{M} isomorphic to $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}'$. With this composition law, \mathfrak{M} is a *group isomorphic to the cohomology group* $H^1(X, \mathcal{O}_X^*)$, where \mathcal{O}_X^* is the subsheaf of \mathcal{O}_X such that $\Gamma(U, \mathcal{O}_X^*)$ is the group of invertible elements of the ring $\Gamma(U, \mathcal{O}_X)$ for each open $U \subset X$ (\mathcal{O}_X^* is therefore a sheaf of *multiplicative* abelian groups).

We will note that for all open $U \subset X$, the group of sections $\Gamma(U, \mathcal{O}_X^*)$ canonically identifies with the *automorphism group* of the $(\mathcal{O}_X|U)$ -module $\mathcal{O}_X|U$, the identification sending a section ε of \mathcal{O}_X^* over U to the automorphism u of $\mathcal{O}_X|U$ such that $u_x(s_x) = \varepsilon_x s_x$ for all $x \in U$ and all $s_x \in \mathcal{O}_X$. Then let $\mathfrak{U} = (U_\lambda)$ be an open cover of X ; the data, for each pair of indices (λ, μ) , of an automorphism $\theta_{\lambda\mu}$ of $\mathcal{O}_X|(U_\lambda \cap U_\mu)$ is the same as giving a *1-cochain* of the cover \mathfrak{U} , with values in \mathcal{O}_X^* , and say that the $\theta_{\lambda\mu}$ satisfy the *gluing condition* (3.3.1), meaning that the corresponding cochain is a *cocycle*. Similarly, the data, for each λ , of an automorphism ω_λ of $\mathcal{O}_X|U_\lambda$ is the same as the data of a *0-cochain* of the cover \mathfrak{U} , with values in \mathcal{O}_X^* , and its *coboundary* corresponds to the family of automorphisms $(\omega_\lambda|U_\lambda \cap U_\mu) \circ (\omega_\mu|U_\lambda \cap U_\mu)^{-1}$. We can send each 1-cocycle of \mathfrak{U} with values in \mathcal{O}_X^* to the element of \mathfrak{M} isomorphic to an invertible \mathcal{O}_X -module obtained by gluing with respect to the family of automorphisms $(\theta_{\lambda\mu})$ corresponding to this cocycle, and to two cohomologous cocycles correspond two equal elements of \mathfrak{M} (3.3.2); in other words, we thus define a map $\phi_{\mathfrak{U}} : H^1(\mathfrak{U}, \mathcal{O}_X^*) \rightarrow \mathfrak{M}$. In addition, if \mathfrak{B} is a second open cover of X , finer than \mathfrak{U} , then the diagram

$$\begin{array}{ccc} H^1(\mathfrak{U}, \mathcal{O}_X^*) & & \mathfrak{M} \\ \downarrow & \searrow \phi_{\mathfrak{U}} & \\ & & \mathfrak{M} \\ & \nearrow \phi_{\mathfrak{B}} & \\ H^1(\mathfrak{B}, \mathcal{O}_X^*) & & \end{array}$$

where the vertical arrow is the canonical homomorphism (G, II, 5.7), is commutative, as a result of (3.3.3). By passing to the inductive limit, we therefore obtain a map $H^1(X, \mathcal{O}_X^*) \rightarrow \mathfrak{M}$, the Čech cohomology group $H^1(X, \mathcal{O}_X^*)$ identifying as we know with the first cohomology group $H^1(X, \mathcal{O}_X^*)$ (G, II, 5.9, Cor. of Thm. 5.9.1). This map is *surjective*: indeed, by definition, for each invertible \mathcal{O}_X -module \mathcal{L} , there is an open cover (U_λ) of X such that \mathcal{L} is obtained by gluing the sheaves

⁸See the book in preparation cited in the introduction.

$\mathcal{O}_X|_{U_\lambda}$ (3.3.1). It is also *injective*, since it suffices to prove for the maps $H^1(U, \mathcal{O}_X) \rightarrow \mathfrak{M}$, and this follows from (3.3.2). It remains to show that the bijection thus defined is a group homomorphism. Given two invertible \mathcal{O}_X -modules \mathcal{L} and \mathcal{L}' , there is an open cover (U_λ) such that $\mathcal{L}|_{U_\lambda}$ and $\mathcal{L}'|_{U_\lambda}$ are isomorphic to $\mathcal{O}_X|_{U_\lambda}$ for each λ ; so there is for each index λ an element a_λ (resp. a'_λ) of $\Gamma(U_\lambda, \mathcal{L})$ (resp. $\Gamma(U_\lambda, \mathcal{L}')$) such that the elements of $\Gamma(U_\lambda, \mathcal{L})$ (resp. $\Gamma(U_\lambda, \mathcal{L}')$) are the $s_\lambda \cdot a_\lambda$ (resp. $s_\lambda \cdot a'_\lambda$), where s_λ varies over $\Gamma(U_\lambda, \mathcal{O}_X)$. The corresponding cocycles $(\varepsilon_{\lambda\mu})$, $(\varepsilon'_{\lambda\mu})$ are such that $s_\lambda \cdot a_\lambda = s_\mu \cdot a_\mu$ (resp. $s_\lambda \cdot a'_\lambda = s_\mu \cdot a'_\mu$) over $U_\lambda \cap U_\mu$ is equivalent to $s_\lambda = \varepsilon_{\lambda\mu} s_\mu$ (resp. $s_\lambda = \varepsilon'_{\lambda\mu} s_\mu$) over $U_\lambda \cap U_\mu$. As the sections of $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}'$ over U_λ are the finite sums of the $s_\lambda s'_\lambda \cdot (a_\lambda \otimes a'_\lambda)$ where s_λ and s'_λ vary over $\Gamma(U_\lambda, \mathcal{O}_X)$, it is clear that the cocycle $(\varepsilon_{\lambda\mu}, \varepsilon'_{\lambda\mu})$ corresponds to $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}'$, which finishes the proof. ⁹

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(5.4.8). Let $f = (\psi, \omega)$ be a morphism $Y \rightarrow X$ of ringed spaces. The functor $f^*(\mathcal{L})$ to the category of free \mathcal{O}_X -modules defines a map (which we still denote f^* by abuse of language) from the set $\mathfrak{M}(X)$ to the set $\mathfrak{M}(Y)$. Second, we have a canonical homomorphism (T, 3.2.2)

$$(5.4.8.1) \quad H^1(X, \mathcal{O}_X^*) \longrightarrow H^1(Y, \mathcal{O}_Y^*).$$

When we canonically identify (5.4.7) $\mathfrak{M}(X)$ and $H^1(X, \mathcal{O}_X^*)$ (resp. $\mathfrak{M}(Y)$ and $H^1(Y, \mathcal{O}_Y^*)$), the homomorphism (5.4.8.1) *identifies with the map* f^* . Indeed, if \mathcal{L} comes from a cocycle $(\varepsilon_{\lambda\mu})$ corresponding to an open cover (U_λ) of X , then it suffices to show that $f^*(\mathcal{L})$ comes from a cocycle whose cohomology class is the image under (5.4.8.1) of $(\varepsilon_{\lambda\mu})$. If $\theta_{\lambda\mu}$ is the automorphism of $\mathcal{O}_X|(U_\lambda \cap U_\mu)$ which corresponds to $\varepsilon_{\lambda\mu}$, then it is clear that $f^*(\mathcal{L})$ is obtained by gluing the $\mathcal{O}_Y|\psi^{-1}(U_\lambda)$ by means of the automorphisms $f^*(\theta_{\lambda\mu})$, and it then suffices to check that these latter automorphisms corresponds to the cocycle $(\omega^\sharp(\varepsilon_{\lambda\mu}))$, which follows immediately from the definitions (we can identify $\varepsilon_{\lambda\mu}$ with its canonical image under ρ (3.7.2), a section of $\psi^*(\mathcal{O}_X^*)$ over $\psi^{-1}(U_\lambda \cap U_\mu)$).

(5.4.9). Let \mathcal{E} and \mathcal{F} be two \mathcal{O}_X -modules, \mathcal{F} assumed to be *locally free*, and let \mathcal{G} be an \mathcal{O}_X -module extension of \mathcal{F} by \mathcal{E} , in other words there exists an exact sequence $0 \rightarrow \mathcal{E} \xrightarrow{i} \mathcal{G} \xrightarrow{p} \mathcal{F} \rightarrow 0$. Then, for each $x \in X$, there exists an open neighbourhood U of x such that $\mathcal{G}|_U$ is isomorphic to the *direct sum* $\mathcal{E}|_U \oplus \mathcal{F}|_U$. We can reduce to the case where $\mathcal{F} = \mathcal{O}_X^n$; let e_i ($1 \leq i \leq n$) be the canonical sections (5.5.5) of \mathcal{O}_X^n ; there then exists an open neighbourhood U of x and n sections s_i of \mathcal{G} over U such that $p(s_i|_U) = e_i|_U$ for $1 \leq i \leq n$. That being so, let f be the homomorphism $\mathcal{F}|_U \rightarrow \mathcal{G}|_U$ defined by the sections $s_i|_U$ (5.1.1). It is immediate that for each open $V \subset U$, and each section $s \in \Gamma(V, \mathcal{G})$ we have $s - f(p(s)) \in \Gamma(V, \mathcal{E})$, hence our assertion.

(5.4.10). Let $f : X \rightarrow Y$ be a morphism of ringed spaces, \mathcal{F} an \mathcal{O}_X -module, and \mathcal{L} a locally free \mathcal{O}_Y -module of finite rank. Then there exists a canonical isomorphism

$$(5.4.10.1) \quad f_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{L} \simeq f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*(\mathcal{L})).$$

Indeed, for each \mathcal{O}_Y -module \mathcal{L} , we have a canonical homomorphism

$$f_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{L} \xrightarrow{1 \otimes \rho} f_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} f_*(f^*(\mathcal{L})) \xrightarrow{\alpha} f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*(\mathcal{L})),$$

ρ the homomorphism (4.4.3.2) and α the homomorphism (4.2.2.1). To show that when \mathcal{L} is locally free, this homomorphism is bijective, it suffices, since the questions is local, to consider the case where $\mathcal{L} = \mathcal{O}_Y^n$; in addition, f_* and f^* commute with finite direct sums, so we can assume $n = 1$, and in this case the proposition follows immediately from the definitions and from the relation $f^*(\mathcal{O}_Y) = \mathcal{O}_X$.

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⁹For a general form of this result, see the book cited in the note on p. 51.

5.5. Sheaves on a locally ringed space

(5.5.1). We say that a ringed space (X, \mathcal{O}_X) is a *locally ringed space* if for each $x \in X$, \mathcal{O}_x is a local ring; these ringed spaces will be by far the most frequent ringed spaces that we will consider in this work. We then denote by \mathfrak{m}_x the *maximal ideal* of \mathcal{O}_x , by $k(x)$ the *residue field* $\mathcal{O}_x/\mathfrak{m}_x$; for each \mathcal{O}_X -module \mathcal{F} , each open set U of X , each point $x \in U$, and each section $f \in \Gamma(U, \mathcal{F})$, we denote by $f(x)$ the *class* of the germ $f_x \in \mathcal{F}_x \bmod \mathfrak{m}_x \mathcal{F}_x$, and we say that this is the *value* of f at the point x . The relation $f(x) = 0$ then means that $f_x \in \mathfrak{m}_x \mathcal{F}_x$; when this is so, we say (by abuse of language) that f is *zero at x* . We will take care not to confuse this relation with $f_x = 0$.

(5.5.2). Let X be a locally ringed space, \mathcal{L} an invertible \mathcal{O}_X -module, and f a section of \mathcal{L} over X . There is then an *equivalence* between the three following properties for a point $x \in X$:

- (a) f_x is a generator of \mathcal{L}_x ;
- (b) $f_x \notin \mathfrak{m}_x \mathcal{L}_x$ (in other words, $f(x) \neq 0$);
- (c) there exists a section g of \mathcal{L}^{-1} over an open neighbourhood V of x such that the canonical image of $f \otimes g$ in $\Gamma(V, \mathcal{O}_X)$ (5.4.3) is the unit section.

Indeed, since the questions is local, we can reduce to the case where $\mathcal{L} = \mathcal{O}_X$; the equivalence of (a) and (b) are then evident, and it is clear that (c) implies (b). Conversely, if $f_x \notin \mathfrak{m}_x$, then f_x is invertible in \mathcal{O}_x , say $f_x g_x = 1_x$. By definition of germs of sections, this means that there exists a neighbourhood V of x and a section g of \mathcal{O}_X over V such that $fg = 1$ in V , hence (c).

It follows immediately from the condition (c) that the set X_f of x satisfying the equivalent conditions (a), (b), (c) is *open* in X ; following the terminology introduced in (5.5.1), this is the set of the x for which f does not vanish.

(5.5.3). Under the hypotheses of (5.5.2), let \mathcal{L}' be a second invertible \mathcal{O}_X -module; then, if $f \in \Gamma(X, \mathcal{L})$, $g \in \Gamma(X, \mathcal{L}')$, we have

$$X_f \cap X_g = X_{f \otimes g}.$$

We can in fact reduce immediately to the case where $\mathcal{L} = \mathcal{L}' = \mathcal{O}_X$ (since the questions is local); as $f \otimes g$ then canonically identifies with the product fg , the proposition is evident.

(5.5.4). Let \mathcal{F} be a locally free \mathcal{O}_X of rank n ; it is immediate that $\wedge^p \mathcal{F}$ is a locally free \mathcal{O}_X -module of rank $\binom{n}{p}$ if $p \leq n$ and 0 if $p > n$, since the question is local and we can reduce to the case where $\mathcal{F} = \mathcal{O}_X^n$; in addition, for each $x \in X$, $(\wedge^p \mathcal{F})_x / \mathfrak{m}_x (\wedge^p \mathcal{F})_x$ is a vector space of dimension $\binom{n}{p}$ over $k(x)$, which canonically identifies with $\wedge^p (\mathcal{F}_x / \mathfrak{m}_x \mathcal{F}_x)$. Let s_1, \dots, s_p be the sections of \mathcal{F} over an open subset U of X , and let $s = s_1 \wedge \dots \wedge s_p$, which is a section of $\wedge^p \mathcal{F}$ over U (4.1.5); we have $s(x) = s_1(x) \wedge \dots \wedge s_p(x)$, and as a result, we say that the $s_1(x), \dots, s_p(x)$ are *linearly dependent* means $s(x) = 0$. We conclude that the *set of the $x \in X$ such that $s_1(x), \dots, s_p(x)$ are linearly independent is open in X* : it suffices in fact, by reducing to the case where $\mathcal{F} = \mathcal{O}_X^n$, to apply (5.5.2) to the section image of s under one of the projections of $\wedge^p \mathcal{F} = \mathcal{O}_X^{\binom{n}{p}}$ to the $\binom{n}{p}$ factors.

In particular, if s_1, \dots, s_n are n sections of \mathcal{F} over U such that $s_1(x), \dots, s_n(x)$ are linearly independent for each point $x \in U$, then the homomorphism $u : \mathcal{O}_X^n|U \rightarrow \mathcal{F}|U$ defined by the s_i (5.1.1) is an *isomorphism*: indeed, we can restrict to the case where $\mathcal{F} = \mathcal{O}_X^n$ and where we canonically identify $\wedge^n \mathcal{F}$ and \mathcal{O}_X ; $s = s_1 \wedge \dots \wedge s_n$ is then an *invertible* section of \mathcal{O}_X over U , and we define an inverse homomorphism for u by means of the Cramer formulas.

(5.5.5). Let \mathcal{E} and \mathcal{F} be two locally free \mathcal{O}_X -modules (of finite rank), and let $u : \mathcal{E} \rightarrow \mathcal{F}$ be a homomorphism. For there to exist a neighbourhood U of $x \in X$ such that $u|U$ is *injective* and that $\mathcal{F}|U$ is the *direct sum of the $u(\mathcal{E})|U$ and of a locally free $(\mathcal{O}_X|U)$ -submodule \mathcal{G}* , it is necessary and sufficient that $u_x : \mathcal{E}_x \rightarrow \mathcal{F}_x$ gives, by passing to quotients, an *injective* homomorphism of vector spaces $\mathcal{E}_x / \mathfrak{m}_x \mathcal{E}_x \rightarrow \mathcal{F}_x / \mathfrak{m}_x \mathcal{F}_x$. The condition is indeed *necessary*, since \mathcal{F}_x is then the direct sum of the free \mathcal{O}_x -modules $u_x(\mathcal{E}_x)$ and \mathcal{G}_x , so $\mathcal{F}_x / \mathfrak{m}_x \mathcal{F}_x$ is the direct sum of $u_x(\mathcal{E}_x) / \mathfrak{m}_x u_x(\mathcal{E}_x)$ and of $\mathcal{G}_x / \mathfrak{m}_x \mathcal{G}_x$. The condition is *sufficient*, since we can reduce to the case where $\mathcal{E} = \mathcal{O}_X^m$; let s_1, \dots, s_m be the images under u of the sections e_i of \mathcal{O}_X^m such that $(e_i)_y$ is equal to the i -th element of the canonical basis of \mathcal{O}_y^m for each $y \in Y$ (*canonical sections of \mathcal{O}_X^m*); by hypothesis, the $s_1(x), \dots, s_m(x)$

are linearly independent, so if \mathcal{F} is of rank n , then there exist $n - m$ sections s_{m+1}, \dots, s_n of \mathcal{F} over a neighbourhood V of x such that the $s_i(x)$ ($1 \leq i \leq n$) form a basis for $\mathcal{F}_x / \mathfrak{m}_x \mathcal{F}_x$. There then exists (5.5.4) a neighbourhood $U \subset V$ of x such that the $s_i(y)$ ($1 \leq i \leq n$) form a basis for $\mathcal{F}_y / \mathfrak{m}_y \mathcal{F}_y$ for each $y \in U$, and we conclude (5.5.4) that there is an isomorphism from $\mathcal{F}|_U$ to $\mathcal{O}_X^n|_U$, sending the $s_i|_U$ ($1 \leq i \leq m$) to the $e_i|_U$, which finishes the proof.

§6. FLATNESS

(6.0). The notion of flatness is due to J.-P. Serre [Ser56]; in the following, we omit the proofs of the results which are presented in the *Algèbre commutative* of N. Bourbaki, to which we refer the reader. We assume that all rings are commutative.¹⁰

If M, N are two A -modules, M' (resp. N') a submodule of M (resp. N), we denote by $\text{Im}(M' \otimes_A N')$ the submodule of $M \otimes_A N$, the image under the canonical map $M' \otimes_A N' \rightarrow M \otimes_A N$. 01 | 55

6.1. Flat modules

(6.1.1). Let M be an A -module. The following conditions are equivalent:

- (a) The functor $M \otimes_A N$ is exact in N on the category of A -modules;
- (b) $\text{Tor}_i^A(M, N) = 0$ for each $i > 0$ and for each A -module N ;
- (c) $\text{Tor}_1^A(M, N) = 0$ for each A -module N .

When M satisfies these conditions, we say that M is a *flat A -module*. It is clear that each free A -module is flat.

For M to be a flat A -module, it suffices that for each ideal \mathfrak{J} of A , of *finite type*, the canonical map $M \otimes_A \mathfrak{J} \rightarrow M \otimes_A A = M$ is *injective*.

(6.1.2). Each inductive limit of flat A -modules is a flat A -module. For a direct sum $\bigoplus_{\lambda \in L} M_\lambda$ of A -modules to be a flat A -module, it is necessary and sufficient that each of the A -modules M_λ is flat. In particular, every projective A -module is flat.

Let $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ be an exact sequence of A -modules, such that M'' is *flat*. Then, for each A -module N , the sequence

$$0 \longrightarrow M' \otimes_A N \longrightarrow M \otimes_A N \longrightarrow M'' \otimes_A N \longrightarrow 0$$

is exact. In addition, for M to be flat, it is necessary and sufficient that M' is (but it can be that M and M' are flat without $M'' = M/M'$ being so).

(6.1.3). Let M be a flat A -module, N any A -module; for two submodules N', N'' of N , we then have

$$\begin{aligned} \text{Im}(M \otimes_A (N' + N'')) &= \text{Im}(M \otimes_A N') + \text{Im}(M \otimes_A N''), \\ \text{Im}(M \otimes_A (N' \cap N'')) &= \text{Im}(M \otimes_A N') \cap \text{Im}(M \otimes_A N'') \end{aligned}$$

(images taken in $M \otimes_A N$).

(6.1.4). Let M and N be two A -modules, M' (resp. N') a submodule of M (resp. N), and suppose that one of the modules $M/M', N/N'$ is flat. Then we have $\text{Im}(M' \otimes_A N') = \text{Im}(M' \otimes_A N) \cap (M \otimes_A N')$ (images in $M \otimes_A N$). In particular, if \mathfrak{J} is an ideal of A and if M/M' is flat, then we have $\mathfrak{J}M' = M' \cap \mathfrak{J}M$.

¹⁰See the exposé cited of N. Bourbaki for the generalization from most of the results to the noncommutative case.

6.2. Change of ring When an additive group M is equipped with multiple modules structures relative to the rings A, B, \dots , we say that M is flat as an A -module, B -module, \dots , we sometimes also say that M is A -flat, B -flat, \dots

(6.2.1). Let A and B be two rings, M an A -module, N an (A, B) -bimodule. If M is flat and if N is B -flat, then $M \otimes_A N$ is B -flat. In particular, if M and N are two flat A -modules, then $M \otimes_A N$ is a flat A -module. If B is an A -algebra and if M is a flat A -module, then the B -module $M_{(B)} = M \otimes_A B$ is flat. Finally, if B is an A -algebra which is flat as an A -module, and if N is a flat B -module, then N is also A -flat.

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(6.2.2). Let A be a ring, B an A -algebra which is flat as an A -module. Let M, N be two A -modules, such that M admits a finite presentation; then the canonical homomorphism

$$(6.2.2.1) \quad \text{Hom}_A(M, N) \otimes_A B \longrightarrow \text{Hom}_B(M \otimes_A B, N \otimes_A B)$$

(sending $u \otimes b$ to the homomorphism $m \otimes b' \mapsto u(m) \otimes b'b$) is an isomorphism.

(6.2.3). Let $(A_\lambda, \phi_{\mu\lambda})$ be a filtered inductive system of rings; let $A = \varinjlim A_\lambda$. On the other hand, for each λ , let M_λ be an A_λ -module, and for $\lambda \leq \mu$ let $\theta_{\mu\lambda} : M_\lambda \rightarrow M_\mu$ be a $\phi_{\mu\lambda}$ -homomorphism, such that $(M_\lambda, \theta_{\mu\lambda})$ is an inductive system; $M = \varinjlim M_\lambda$ is then an A -module. This being so, if for each λ , M_λ is a flat A_λ -module, then M is a flat A -module. Indeed, let \mathfrak{J} be an ideal of finite type of A ; by definition of the inductive limit, there exists an index λ and an ideal \mathfrak{J}_λ of A_λ such that $\mathfrak{J} = \mathfrak{J}_\lambda A$. If we put $\mathfrak{J}'_\mu = \mathfrak{J}_\lambda A_\mu$ for $\mu \geq \lambda$, we also have $\mathfrak{J} = \varinjlim \mathfrak{J}'_\mu$ (where μ varies over the indices $\geq \lambda$), hence (the functor \varinjlim being exact and commuting with tensor products)

$$M \otimes_A \mathfrak{J} = \varinjlim (M_\mu \otimes_{A_\mu} \mathfrak{J}'_\mu) = \varinjlim \mathfrak{J}'_\mu M_\mu = \mathfrak{J}M.$$

6.3. Local nature of flatness

(6.3.1). If A is a ring, S a multiplicative subset of A , $S^{-1}A$ is a flat A -module. Indeed, for each A -module N , $N \otimes_A S^{-1}A$ identifies with $S^{-1}N$ (1.2.5) and we know (1.3.2) that $S^{-1}N$ is an exact functor in N .

If now M is a flat A -module, $S^{-1}M = M \otimes_A S^{-1}A$ is a flat $S^{-1}A$ -module (6.2.1), so it is also A -flat according to the above and from (6.2.1). In particular, if P is an $S^{-1}A$ -module, we can consider it as an A -module isomorphic to $S^{-1}P$; for P to be A -flat, it is necessary and sufficient that it is $S^{-1}A$ -flat.

(6.3.2). Let A be a ring, B an A -algebra, and T a multiplicative subset of B . If P is a B -module which is A -flat, $T^{-1}P$ is A -flat. Indeed, for each A -module N , we have $(T^{-1}P) \otimes_A N = (T^{-1}B \otimes_B P) \otimes_A N = T^{-1}B \otimes_B (P \otimes_A N) = T^{-1}(P \otimes_A N)$; $T^{-1}(P \otimes_A N)$ is an exact functor in N , being the composition of the two exact functors $P \otimes_A N$ (in N) and $T^{-1}Q$ (in Q). If S is a multiplicative subset of A such that its image in B is contained in T , then $T^{-1}P$ is equal to $S^{-1}(T^{-1}P)$, so it is also $S^{-1}A$ -flat according to (6.3.1).

(6.3.3). Let $\phi : A \rightarrow B$ be a ring homomorphism, M a B -module. The following properties are equivalent:

- (a) M is a flat A -module.
- (b) For each maximal ideal \mathfrak{n} of B , $M_{\mathfrak{n}}$ is a flat A -module.
- (c) For each maximal ideal \mathfrak{n} of B , by setting $\mathfrak{m} = \phi^{-1}(\mathfrak{n})$, $M_{\mathfrak{n}}$ is a flat $A_{\mathfrak{m}}$ -module.

Indeed, as $M_{\mathfrak{n}} = (M_{\mathfrak{n}})_{\mathfrak{m}}$, the equivalence of (b) and (c) follows from (6.3.1), and the fact that (a) implies (b) is a particular case of (6.3.2). It remains to see that (b) implies (a), that is to say, that for each injective homomorphism $u : N' \rightarrow N$ of A -modules, the homomorphism $v = 1 \otimes u : M \otimes_A N' \rightarrow M \otimes_A N$ is injective. We have that v is also a homomorphism of B -modules, and we know that for it to be injective, it suffices that for each maximal ideal \mathfrak{n} of B , $v_{\mathfrak{n}} : (M \otimes_A N')_{\mathfrak{n}} \rightarrow (M \otimes_A N)_{\mathfrak{n}}$ is injective. But as

$$(M \otimes_A N)_{\mathfrak{n}} = B_{\mathfrak{n}} \otimes_B (M \otimes_A N) = M_{\mathfrak{n}} \otimes_A N,$$

$v_{\mathfrak{n}}$ is none other than the homomorphism $1 \otimes u : M_{\mathfrak{n}} \otimes_A N' \rightarrow M_{\mathfrak{n}} \otimes_A N$, which is injective since $M_{\mathfrak{n}}$ is A -flat.

In particular (by taking $B = A$), for an A -module M to be flat, it is necessary and sufficient that $M_{\mathfrak{m}}$ is $A_{\mathfrak{m}}$ -flat for each maximal ideal \mathfrak{m} of A .

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(6.3.4). Let M be an A -module; if M is flat, and if $f \in A$ does not divide 0 in A , f does not kill any element $\neq 0$ in M , since the homomorphism $m \mapsto f \cdot m$ is expressed as $1 \otimes u$, where u is the multiplication $a \mapsto f \cdot a$ on A and M is identified with $M \otimes_A A$; if u is injective, it is the same for $1 \otimes u$ since M is flat. In particular, if A is *integral*, M is *torsion-free*.

Conversely, suppose that A is integral, M is torsion-free, and suppose that for each maximal ideal \mathfrak{m} of A , $A_{\mathfrak{m}}$ is a *discrete valuation ring*; then M is A -flat. Indeed, it suffices (6.3.3) to prove that $M_{\mathfrak{m}}$ is $A_{\mathfrak{m}}$ -flat, and we can therefore suppose that A is already a discrete valuation ring. But as M is the inductive limit of its submodules of finite type, and these latter submodules are torsion-free, we can in addition reduce to the case where M is of finite type (6.1.2). The proposition follows in this case from that M is a free A -module.

In particular, if A is an *integral ring*, $\phi : A \rightarrow B$ a ring homomorphism making B a *flat* A -module and $\neq \{0\}$, ϕ is necessarily *injective*. Conversely, if B is integral, A a subring of B , and if for each maximal ideal \mathfrak{m} of A , $A_{\mathfrak{m}}$ is a discrete valuation ring, then B is A -flat.

6.4. Faithfully flat modules

(6.4.1). For an A -module M , the following four properties are equivalent:

- (a) For a sequence $N' \rightarrow N \rightarrow N''$ of A -modules to be exact, it is necessary and sufficient that the sequence $M \otimes_A N' \rightarrow M \otimes_A N \rightarrow M \otimes_A N''$ is exact;
- (b) M is flat for each A -module N , the relation $M \otimes_A N = 0$ implies $N = 0$;
- (c) M is flat for each homomorphism $v : N \rightarrow N'$ of A -modules, the relation $1_M \otimes v = 0$, 1_M being the identity automorphism of M ;
- (d) M is flat for each maximal ideal \mathfrak{m} of A , $\mathfrak{m}M \neq M$.

When M satisfies these conditions, we say that M is a *faithfully flat* A -module; M is then necessarily a *faithful* module. In addition, if $u : N \rightarrow N'$ is a homomorphism of A -modules, then for u to be injective (resp. surjective, bijective), it is necessary and sufficient that $1 \otimes u : M \otimes_A N \rightarrow M \otimes_A N'$ is so.

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(6.4.2). A free module $\neq \{0\}$ is faithfully flat; it is the same for the direct sum of a flat module and a faithfully flat module. If S is a multiplicative subset of A , then $S^{-1}A$ is a faithfully flat A -module if S consists of invertible elements (so $S^{-1}A = A$).

(6.4.3). Let $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ be an exact sequence of A -modules; if M' and M'' are flat, and if one of the two is faithfully flat, then M is also faithfully flat.

(6.4.4). Let A and B be two rings, M an A -module, N an (A, B) -bimodule. If M is faithfully flat and if N is a faithfully flat B -module, then $M \otimes_A N$ is a faithfully flat B -module. In particular, if M and N are two faithfully flat A -modules, then so is $M \otimes_A N$. If B is an A -algebra and if M is a faithfully flat A -module, the B -module $M_{(B)}$ is faithfully flat.

(6.4.5). If M is a faithfully flat A -modules and if S is a multiplicative subset of A , $S^{-1}M$ is a faithfully flat $S^{-1}A$ -module, since $S^{-1}M = M \otimes_A (S^{-1}A)$ (6.4.4). Conversely, if for each maximal ideal \mathfrak{m} of A , $M_{\mathfrak{m}}$ is a faithfully flat $A_{\mathfrak{m}}$ -module, then M is a faithfully flat A -module, since M is A -flat (6.3.3), and we have

$$M_{\mathfrak{m}}/\mathfrak{m}M_{\mathfrak{m}} = (M \otimes_A A_{\mathfrak{m}}) \otimes_{A_{\mathfrak{m}}} (A_{\mathfrak{m}}/\mathfrak{m}A_{\mathfrak{m}}) = M \otimes_A (A/\mathfrak{m}) = M/\mathfrak{m}M,$$

so the hypotheses imply that $M/\mathfrak{m}M \neq 0$ for each maximal ideal \mathfrak{m} of A , which proves our assertion (6.4.1).

6.5. Restriction of scalars

(6.5.1). Let A be a ring, $\phi : A \rightarrow B$ a ring homomorphism making B an A -algebra. Suppose that there exists a B -module N which is a *faithfully flat* A -module. Then, for each A -module M , the homomorphism $x \mapsto 1 \otimes x$ from M to $B \otimes_A M = M_{(B)}$ is *injective*. In particular, ϕ is injective; for each ideal \mathfrak{a} of A , we have $\phi^{-1}(\mathfrak{a}B) = \mathfrak{a}$; for each maximal (resp. prime) ideal \mathfrak{m} of A , there exists a maximal (resp. prime) ideal \mathfrak{n} of B such that $\phi^{-1}(\mathfrak{n}) = \mathfrak{m}$.

(6.5.2). When the conditions of (6.5.1) are satisfied, we identify A with the subring of B by ϕ and more generally, for each A -module M , we identify M with an A -submodule of $M_{(B)}$. We note that if B is also *Noetherian*, then so is A , since the map $\mathfrak{a} \mapsto \mathfrak{a}B$ is an increasing injection from the set of ideals of A to the set of ideals of B ; the existence of an infinite strictly increasing sequence of ideals of A thus implies the existence of an analogous sequence of ideals of B .

6.6. Faithfully flat rings

(6.6.1). Let $\phi : A \rightarrow B$ be a ring homomorphism making B an A -algebra. The following five properties are equivalent:

- (a) B is a faithfully flat A -module (in other words, $M_{(B)}$ is an *exact* and *faithful* functor in M).
- (b) The homomorphism ϕ is injective and the A -module $B/\phi(A)$ is flat.
- (c) The A -module B is flat (in other words, the functor $M_{(B)}$ is *exact*), and for each A -module M , the homomorphism $x \mapsto 1 \otimes x$ from M to $M_{(B)}$ is injective.
- (d) The A -module B is flat and for each ideal \mathfrak{a} of A , we have $\phi^{-1}(\mathfrak{a}B) = \mathfrak{a}$.
- (e) The A -module B is flat and for each maximal ideal \mathfrak{m} of A , there exists a maximal ideal \mathfrak{n} of B such that $\phi^{-1}(\mathfrak{n}) = \mathfrak{m}$.

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When these conditions are satisfied, we identify A with a subring of B .

(6.6.2). Let A be a *local* ring, \mathfrak{m} its maximal ideal, and B an A -algebra such that $\mathfrak{m}B \neq B$ (which is so when for example B is a local ring and $A \rightarrow B$ is a *local* homomorphism). If B is a *flat* A -module, B is a *faithfully flat* A -module. Indeed, this follows from (6.4.1, (d)). Under the indicated conditions, we thus see that if B is *Noetherian*, then so too is A (6.5.2).

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(6.6.3). Let B be an A -algebra which is a faithfully flat A -module. For each A -module M and each A -submodule M' of M , we have (by identifying M with an A -submodule of $M_{(B)}$) $M' = M \cap M'_{(B)}$. For M to be a flat (resp. faithfully flat) A -module, it is necessary and sufficient that $M_{(B)}$ is a flat (resp. faithfully flat) B -module.

(6.6.4). Let B be an A -algebra, N a faithfully flat B -module. For B to be a flat (resp. faithfully flat) A -module, it is necessary and sufficient that N is.

In particular, let C be a B -algebra; if the ring C is faithfully flat over B and B is faithfully flat over A , then C is faithfully flat over A ; if C is faithfully flat over B and over A , then B is faithfully flat over A .

6.7. Flat morphisms of ringed spaces

(6.7.1). Let $f : X \rightarrow Y$ be a morphism of ringed spaces, and let \mathcal{F} be an \mathcal{O}_X -module. We say that \mathcal{F} is *f-flat* (or *Y-flat* when there is no chance of confusion with f) at a point $x \in X$ if \mathcal{F}_x is a flat $\mathcal{O}_{f(x)}$ -module; we say that \mathcal{F} is *f-flat over* $y \in Y$ if \mathcal{F} is *f-flat* for all the points $x \in f^{-1}(y)$; we say that \mathcal{F} is *f-flat* if \mathcal{F} is *f-flat* at all the points of X . We say that the morphism f is *flat at* $x \in X$ (resp. *flat over* $y \in Y$, resp. *flat*) if \mathcal{O}_X is *f-flat at* x (resp. *f-flat over* y , resp. *f-flat*). If f is a flat morphism, we then say that X is *flat over* Y , or *Y-flat*.

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(6.7.2). With the notation of (6.7.1), if \mathcal{F} is *f-flat at* x , for each open neighbourhood U of $y = f(x)$, the functor $(f^*(\mathcal{G}) \otimes_{\mathcal{O}_X} \mathcal{F})_x$ in \mathcal{G} is *exact* on the category of $(\mathcal{O}_Y|U)$ -modules; indeed, this stalk canonically identifies with $\mathcal{G}_y \otimes_{\mathcal{O}_y} \mathcal{F}_x$, and our assertion follows from the definition. In particular, if f is a *flat* morphism, the functor f^* is *exact* on the category of \mathcal{O}_Y -modules.

(6.7.3). Conversely, suppose the sheaf of rings \mathcal{O}_Y is *coherent*, and suppose that for *each* open neighbourhood U of y , the functor $(f^*(\mathcal{G}) \otimes_{\mathcal{O}_X} \mathcal{F})_x$ is *exact* in \mathcal{G} on the category of *coherent* $(\mathcal{O}_Y|U)$ -modules. Then \mathcal{F} is *f-flat at* x . In fact, it suffices to prove that for each ideal of finite type \mathfrak{J} of \mathcal{O}_y ,

the canonical homomorphism $\mathfrak{J} \otimes_{\mathcal{O}_y} \mathcal{F}_x \rightarrow \mathcal{F}_x$ is injective (6.1.1). We know (5.3.8) that there then exists an open neighbourhood U of y and a coherent sheaf of ideals \mathcal{I} of $\mathcal{O}_Y|U$ such that $\mathcal{I}_y = \mathfrak{J}$, hence the conclusion. 01 | 60

(6.7.4). The results of (6.1) for flat modules are immediately translated into propositions for sheaves with are f -flat at a point:

If $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ is an exact sequence of \mathcal{O}_X -modules and if \mathcal{F}'' is f -flat at a point $x \in X$, then, for each open neighbourhood U of $y = f(x)$ and each $(\mathcal{O}_Y|U)$ -module \mathcal{G} , the sequence

$$0 \longrightarrow (f^*(\mathcal{G}) \otimes_{\mathcal{O}_X} \mathcal{F}')_x \longrightarrow (f^*(\mathcal{G}) \otimes_{\mathcal{O}_X} \mathcal{F})_x \longrightarrow (f^*(\mathcal{G}) \otimes_{\mathcal{O}_X} \mathcal{F}'')_x \longrightarrow 0$$

is exact. For \mathcal{F} to be f -flat at x , it is necessary and sufficient that \mathcal{F}' is. We have similar statements for the corresponding notions of a f -flat \mathcal{O}_X -modules over $y \in Y$, or of a f -flat \mathcal{O}_X -module.

(6.7.5). Let $f : X \rightarrow Y, g : Y \rightarrow Z$ be two morphisms of ringed spaces; let $x \in X, y = f(x)$, and \mathcal{F} be an \mathcal{O}_X -module. If \mathcal{F} is f -flat at the point x and if the morphism g is flat at the point y , then \mathcal{F} is $(g \circ f)$ -flat at x (6.2.1). In particular, if f and g are flat morphisms, then $g \circ f$ is flat.

(6.7.6). Let X, Y be two ringed spaces, $f : X \rightarrow Y$ a flat morphism. Then the canonical homomorphism of bifunctors (4.4.6)

$$(6.7.6.1) \quad f^*(\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{G})) \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(f^*(\mathcal{F}), f^*(\mathcal{G}))$$

is an isomorphism when \mathcal{F} admits a finite presentation (5.2.5).

Indeed, since the questions is local, we can assume that there exists an exact sequence $\mathcal{O}_Y^m \rightarrow \mathcal{O}_Y^n \rightarrow \mathcal{F} \rightarrow 0$. The two sides of (6.7.6.1) are right exact functors in \mathcal{F} according to the hypotheses on f ; we then have reduced to proving the proposition in the case where $\mathcal{F} = \mathcal{O}_Y$, in which the result is trivial.

(6.7.8). We say that a morphism $f : X \rightarrow Y$ of ringed spaces is *faithfully flat* if f is *surjective* and if, for each $x \in X, \mathcal{O}_x$ is a *faithfully flat* $\mathcal{O}_{f(x)}$ -module. When X and Y are locally ringed spaces (5.5.1), it is equivalent to say that the morphism f is *surjective* and *flat* (6.6.2). When f is faithfully flat, f^* is an *exact* and *faithful* functor on the category of \mathcal{O}_Y -modules (6.6.1, a), and for an \mathcal{O}_Y -module \mathcal{G} to be Y -flat, it is necessary and sufficient that $f^*(\mathcal{G})$ is (6.6.3).

§7. ADIC RINGS

7.1. Admissible rings

(7.1.1). Recall that in a topological ring A (not necessarily separated), we say that an element x is *topologically nilpotent* if 0 is a limit of the sequence $(x^n)_{n \geq 0}$. We say that a topological ring A is *linearly topologized* if there exists a fundamental system of neighbourhoods of 0 in A of (necessarily *open*) *ideals*.

Definition (7.1.2). — In a linearly topologized ring A , we say that an ideal \mathfrak{J} is an *ideal of definition* if \mathfrak{J} is open and if, for each neighbourhood V of 0, there exists a integer $n > 0$ such that $\mathfrak{J}^n \subset V$ (which we express, by abuse of language, by saying that the sequence (\mathfrak{J}^n) tends to 0). We say that a linearly topologized ring A is *preadmissible* if there exists in A an ideal of definition; we say that A is *admissible* if it is preadmissible and if in addition it is separated and complete. 01 | 61

It is clear that if \mathfrak{J} is an ideal of definition, \mathfrak{L} an open ideal of A , then $\mathfrak{J} \cap \mathfrak{L}$ is also an ideal of definition; the ideals of definition of a preadmissible ring A thus form a *fundamental system of neighbourhoods* of 0.

Lemma (7.1.3). — *Let A be a linearly topologized ring.*

- (i) *For $x \in A$ to be topologically nilpotent, it is necessary and sufficient that for each open ideal \mathfrak{J} of A , the canonical image of x in A/\mathfrak{J} is nilpotent. The set \mathfrak{T} of topologically nilpotent elements of A is an ideal.*
- (ii) *Suppose that in addition A is preadmissible, and let \mathfrak{J} be an ideal of definition for A . For $x \in A$ to be topologically nilpotent, it is necessary and sufficient that its canonical image in A/\mathfrak{J} is nilpotent; the ideal \mathfrak{T} is the inverse image of the nilradical of A/\mathfrak{J} and is thus open.*

Proof. (i) follows immediately from the definitions. To prove (ii), it suffices to note that for each neighbourhood V of 0 in A , there exists an $n > 0$ such that $\mathfrak{J}^n \subset V$; if $x \in A$ is such that $x^m \in \mathfrak{J}$, we have $x^{mq} \in V$ for $q \geq n$, so x is topologically nilpotent. \square

Proposition (7.1.4). — *Let A be a preadmissible ring, \mathfrak{J} an ideal of definition for A .*

- (i) *For an ideal \mathfrak{J}' of A to be contained in an ideal of definition, it is necessary and sufficient that there exists an integer $n > 0$ such that $\mathfrak{J}'^n \subset \mathfrak{J}$.*
- (ii) *For an $x \in A$ to be contained in an ideal of definition, it is necessary and sufficient that it is topologically nilpotent.*

Proof.

- (i) If $\mathfrak{J}'^m \subset \mathfrak{J}$, then for each open neighbourhood V of 0 in A , there exists an m such that $\mathfrak{J}'^m \subset V$, thus $\mathfrak{J}'^{mm} \subset V$.
- (ii) The condition is evidently necessary; it is sufficient, since if it satisfied, then there exists an n such that $x^n \in \mathfrak{J}$, so $\mathfrak{J}' = \mathfrak{J} + Ax$ is an ideal of definition, because it is open, and $\mathfrak{J}'^n \subset \mathfrak{J}$. \square

Corollary (7.1.5). — *In a preadmissible ring A , an open prime ideal contains all the ideals of definition.*

Corollary (7.1.6). — *The notation and hypotheses being that of (7.1.4), the following properties of an ideal \mathfrak{J}_0 of A are equivalent:*

- (a) \mathfrak{J}_0 is the largest ideal of definition of A ;
- (b) \mathfrak{J}_0 is a maximal ideal of definition;
- (c) \mathfrak{J}_0 is an ideal of definition such that the ring A/\mathfrak{J}_0 is reduced.

For there to exist an ideal \mathfrak{J}_0 to have these properties, it is necessary and sufficient that the nilradical of A/\mathfrak{J} to be nilpotent; \mathfrak{J}_0 is then equal to the ideal \mathfrak{T} of topologically nilpotent elements of A .

Proof. It is clear that (a) implies (b), and (b) implies (c) according to (7.1.4, ii), and (7.1.3, ii); for the same reason, (c) implies (a). The latter assertion follows from (7.1.4, i) and (7.1.3, ii). \square

When $\mathfrak{T}/\mathfrak{J}$, the nilradical of A/\mathfrak{J} , is nilpotent, and we denote by A_{red} the (reduced) quotient ring A/\mathfrak{T} .

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Corollary (7.1.7). — *A preadmissible Noetherian ring admits a largest ideal of definition.*

Corollary (7.1.8). — *If a preadmissible ring A is such that, for an ideal of definition \mathfrak{J} , the powers \mathfrak{J}^n ($n > 0$) form a fundamental system of neighbourhoods of 0, it is the same for the powers \mathfrak{J}'^n for each ideal of definition \mathfrak{J}' of A .*

Definition (7.1.9). — We say that a preadmissible ring A is *preadic* if there exists an ideal of definition \mathfrak{J} for A such that the \mathfrak{J}^n form a fundamental system of neighbourhoods of 0 in A (or equivalently, such that the \mathfrak{J}^n are open). We call a ring *adic* if it is a separated and complete preadic ring.

If \mathfrak{J} is an ideal of definition for a preadic (resp. adic) ring A , we say that A is a \mathfrak{J} -preadic (resp. \mathfrak{J} -adic) ring, and that its topology is the \mathfrak{J} -preadic (resp. \mathfrak{J} -adic) topology. More generally, if M is an A -module, the topology on M having for a fundamental system of neighbourhoods of 0 the submodules $\mathfrak{J}^n M$ is called the \mathfrak{J} -preadic (resp. \mathfrak{J} -adic) topology. According to (7.1.8), these topologies are independent of the choice of the ideal of definition \mathfrak{J} .

Proposition (7.1.10). — *Let A be an admissible ring, \mathfrak{J} an ideal of definition for A . Then \mathfrak{J} is contained in the radical of A .*

This statement is equivalent to any of the following corollaries:

Corollary (7.1.11). — *For each $x \in \mathfrak{J}$, $1 + x$ is invertible in A .*

Corollary (7.1.12). — *For $f \in A$ to be invertible in A , it is necessary and sufficient that its canonical image in A/\mathfrak{J} is invertible in A/\mathfrak{J} .*

Corollary (7.1.13). — *For each A -module M of finite type, the relation $M = \mathfrak{J}M$ (equivalent to $M \otimes_A (A/\mathfrak{J}) = 0$) implies that $M = 0$.*

Corollary (7.1.14). — *Let $u : M \rightarrow N$ be a homomorphism of A -modules, N being of finite type; for u to be surjective, it is necessary and sufficient that $u \otimes 1 : M \otimes_A (A/\mathfrak{J}) \rightarrow N \otimes_A (A/\mathfrak{J})$ is.*

Proof. The equivalence of (7.1.10) and (7.1.11) follows from Bourbaki, *Alg.*, chap. VIII, §6, no. 3, th. 1, and the equivalence of (7.1.10) and (7.1.10) and (7.1.13) follows from *loc. cit.*, th. 2; the fact that (7.1.10) implies (7.1.14) follows from *loc. cit.*, cor. 4 of the prop. 6; on the other hand, (7.1.14) implies (7.1.13) by applying the zero homomorphism. Finally, (7.1.10) implies that if f is invertible in A/\mathfrak{J} , then f is not contained in any maximal ideal of A , thus f is invertible in A , in other words, (7.1.10) implies (7.1.12); conversely, (7.1.12) implies (7.1.11).

It therefore remains to prove (7.1.11). Now as A is separated and complete, and the sequence (\mathfrak{J}^n) tends to 0, it is immediate that the series $\sum_{n=0}^{\infty} (-1)^n x^n$ is convergent in A , and that if y is its sum, then we have $y(1+x) = 1$. □

7.2. Adic rings and projective limits

(7.2.1). Each projective limit of discrete rings is evidently a linearly topologized ring, separated and compact. Conversely, let A be a linearly topologized ring, and let (\mathfrak{J}_λ) be a fundamental system of open neighbourhoods of 0 in A consisting of ideals. The canonical maps $\phi_\lambda : A \rightarrow A/\mathfrak{J}_\lambda$ form a projective system of continuous representations and therefore define a continuous representation $\phi : A \rightarrow \varprojlim A/\mathfrak{J}_\lambda$; if A is separated, then ϕ is a topological isomorphism from A to an everywhere-dense subring of $\varprojlim A/\mathfrak{J}_\lambda$; if in addition A is complete, then ϕ is a topological isomorphism from A to $\varprojlim A/\mathfrak{J}_\lambda$. 01 | 63

Lemma (7.2.2). — *For a linearly topologized ring to be admissible, it is necessary and sufficient that it is isomorphic to a projective limit $A = \varprojlim A_\lambda$, where $(A_\lambda, \mu_{\lambda\mu})$ is a projective limit of discrete rings having for the set of indices a filtered ordered (by \leq) L which admits a smallest element denoted 0 and satisfies the following conditions: 1st. the $u_\lambda : A \rightarrow A_\lambda$ are surjective; 2nd. the kernel \mathfrak{J}_λ of $u_{0\lambda} : A_\lambda \rightarrow A_0$ is nilpotent. When this is so, the kernel \mathfrak{J} of $u_0 : A \rightarrow A_0$ is equal to $\varprojlim \mathfrak{J}_\lambda$.*

Proof. The necessity of the condition follows from (7.2.1), by choosing (\mathfrak{J}_λ) to be a fundamental system of neighbourhoods of 0 consisting of ideals of definitions contained in an ideal of definition \mathfrak{J}_0 and by applying (7.1.4, i). The converse follows from the definition of the projective limit and from (7.2.1), and the latter assertion is immediate. □

(7.2.3). Let A be an admissible topological ring, \mathfrak{J} an ideal of A contained in an ideal of definition (in other words (7.1.4) such that (\mathfrak{J}^n) tends to 0); we can consider on A the ring topology having for a fundamental system of neighbourhoods of 0 the powers \mathfrak{J}^n ($n > 0$); we call again this the \mathfrak{J} -preadic topology. The hypothesis that A is admissible implies that $\bigcup_n \mathfrak{J}^n = 0$, therefore the \mathfrak{J} -preadic topology on A is separated; let $\hat{A} = \varprojlim A/\mathfrak{J}^n$ be the completion of A for this topology (where the A/\mathfrak{J}^n are equipped with the discrete topology), and denote by u the (not necessarily continuous) ring homomorphism $A \rightarrow \hat{A}$, the projective limit of the sequence of homomorphisms $u_n : A \rightarrow A/\mathfrak{J}^n$. On the other hand, the \mathfrak{J} -preadic topology on A is finer than the given topology \mathcal{T} on A ; as A is separated and complete for \mathcal{T} , we can extend by continuity the identity map of A (equipped with the \mathfrak{J} -preadic topology) to A equipped with \mathcal{T} ; this gives a continuous representation $v : \hat{A} \rightarrow A$.

Proposition (7.2.4). — *If A is an admissible ring and \mathfrak{J} is contained in an ideal of definition of A , then A is separated and complete for the \mathfrak{J} -preadic topology.*

Proof. With the notation of (7.2.3), it is immediate that $v \circ u$ is the identity map of A . On the other hand, $u_n \circ v : \hat{A} \rightarrow A/\mathfrak{J}^n$ is the extension by continuity (for the \mathfrak{J} -preadic topology on A and the discrete topology on A/\mathfrak{J}^n) of the canonical map u_n ; in other words, it is the canonical map from $\hat{A} = \varprojlim_k A/\mathfrak{J}^k$ to A/\mathfrak{J}^n ; $u \circ v$ is therefore the projective limit of this sequence of maps, which is by definition the identity map of \hat{A} ; this proves the proposition. □

Corollary (7.2.5). — *Under the hypotheses of (7.2.3), the following conditions are equivalent:*

- (a) the homomorphism u is continuous;
- (b) the homomorphism v is bicontinuous;
- (c) A is a \mathfrak{J} -adic ring.

Corollary (7.2.6). — *Let A be an admissible ring A , \mathfrak{J} an ideal of definition for A . For A to be Noetherian, it is necessary and sufficient for A/\mathfrak{J} to be Noetherian and for $\mathfrak{J}/\mathfrak{J}^2$ to be an A/\mathfrak{J} -module of finite type.*

These conditions are evidently necessary. Conversely, suppose the conditions are satisfied; as according to (7.2.4) A is complete for the \mathfrak{J} -preadic topology, for it to be Noetherian, it is necessary and sufficient that the associated graded ring $\text{grad}(A)$ (for the filtration on the \mathfrak{J}^n) is ([CC, p. 18–07, th. 4]). Now, let a_1, \dots, a_n be the elements of \mathfrak{J} whose classes mod. \mathfrak{J}^2 are the generators of $\mathfrak{J}/\mathfrak{J}^2$ as a A/\mathfrak{J} -module. It is immediate by induction that the classes mod. \mathfrak{J}^{m+1} of the monomials of total degree m in the a_i ($1 \leq i \leq n$) form a system of generators for the A/\mathfrak{J} -module $\mathfrak{J}^m/\mathfrak{J}^{m+1}$. We conclude that $\text{grad}(A)$ is a ring isomorphic to a quotient of $(A/\mathfrak{J})[T_1, \dots, T_n]$ (T_i indeterminates), which finishes the proof.

Proposition (7.2.7). — *Let (A_i, u_{ij}) be a projective system ($i \in \mathbf{N}$) of discrete rings, and for each integer i , let \mathfrak{J}_i be the kernel in A_i of the homomorphism $u_{0i} : A_i \rightarrow A_0$. We suppose that:*

- (a) *For $i \leq j$, u_{ij} is surjective and its kernel is \mathfrak{J}_j^{i+1} (therefore A_i is isomorphic to A_j/\mathfrak{J}_j^{i+1}).*
- (b) *$\mathfrak{J}_1/\mathfrak{J}_1^2 (= \mathfrak{J}_1)$ is a module of finite type over $A_0 = A_1/\mathfrak{J}_1$.*

Let $A = \varprojlim_i A_i$, and for each integer $n \geq 0$, let u_n be the canonical homomorphism $A \rightarrow A_n$, and let $\mathfrak{J}^{(n+1)} \subset A$ be its kernel. Then we have these conditions:

- (i) *A is an adic ring, having $\mathfrak{J} = \mathfrak{J}^{(1)}$ for an ideal of definition.*
- (ii) *We have $\mathfrak{J}^{(n)} = \mathfrak{J}^n$ for each $n \geq 1$.*
- (iii) *$\mathfrak{J}/\mathfrak{J}^2$ is isomorphic to $\mathfrak{J}_1 = \mathfrak{J}_1/\mathfrak{J}_1^2$, and as a result is a module of finite type over $A_0 = A/\mathfrak{J}$.*

Proof. The hypothesis of surjectivity on the u_{ij} implies that u_n is surjective; in addition, the hypothesis (a) implies that $\mathfrak{J}_j^{j+1} = 0$, therefore A is an admissible ring (7.2.2); by definition, the $\mathfrak{J}^{(n)}$ form a fundamental system of neighbourhoods of 0 in A , so (ii) implies (i). In addition, we have $\mathfrak{J} = \varprojlim_i \mathfrak{J}_i$ and the maps $\mathfrak{J} \rightarrow \mathfrak{J}_i$ are surjective, so (ii) implies (iii), and it remains to prove (ii). By definition, $\mathfrak{J}^{(n)}$ consists of the elements $(x_k)_{k \geq 0}$ of A such that $x_k = 0$ for $k < n$, therefore $\mathfrak{J}^{(n)}\mathfrak{J}^{(m)} \subset \mathfrak{J}^{(n+m)}$, in other words the $\mathfrak{J}^{(n)}$ form a filtration of A . On the other hand, $\mathfrak{J}^{(n)}/\mathfrak{J}^{(n+1)}$ is isomorphic to the projection from $\mathfrak{J}^{(n)}$ to A_n ; as $\mathfrak{J}^{(n)} = \varprojlim_{i > n} \mathfrak{J}_i^n$, this projection is none other than \mathfrak{J}_n^n , which is a module over $A_0 = A_n/\mathfrak{J}_n$. Now let $a_j = (a_{jk})_{k \geq 0}$ be r elements of $\mathfrak{J} = \mathfrak{J}^{(1)}$ such that a_{11}, \dots, a_{r1} form a system of generators for \mathfrak{J}_1 over A_0 ; we will see that the set S_n of monomials of total degree n and the a_j generate the ideal $\mathfrak{J}^{(n)}$ of A . As $\mathfrak{J}_i^{i+1} = 0$, it is clear first of all that $S_n \subset \mathfrak{J}^{(n)}$; since A is complete for the filtration $(\mathfrak{J}^{(m)})$, it suffices to prove that the set \bar{S}_n of classes mod. $\mathfrak{J}^{(n+1)}$ of elements of S_n generate the graded module $\text{grad}(\mathfrak{J}^{(n)})$ over the graded ring $\text{grad}(A)$ for the above filtration ([CC, p. 18–06, lemme]); according to the definition of the multiplication on $\text{grad}(A)$, it suffices to prove that for each m , \bar{S}_m is a system of generators for the A_0 -module $\mathfrak{J}^{(m)}/\mathfrak{J}^{(m+1)}$, or that \mathfrak{J}_m^m is generated by the monomials of degree m in the a_{jm} ($1 \leq j \leq r$). For this, it remains to show that \mathfrak{J}_m is generated (as an A_m -module) by the monomials of degree $\leq m$ with respect to a_{jm} ; the proposition being evident by definition for $m = 1$, we argue by induction on m , and let \mathfrak{J}'_m be the A_m -submodule of \mathfrak{J}_m generated by these monomials. The relation $\mathfrak{J}_{m-1} = \mathfrak{J}_m/\mathfrak{J}_m^m$ and the induction hypothesis prove that $\mathfrak{J}_m = \mathfrak{J}'_m + \mathfrak{J}_m^m$, hence, since $\mathfrak{J}_m^{m+1} = 0$, we have $\mathfrak{J}_m^m = \mathfrak{J}'_m{}^m$, and finally $\mathfrak{J}_m = \mathfrak{J}'_m$. \square

Corollary (7.2.8). — *Under the conditions of Proposition (7.2.7), for A to be Noetherian, it is necessary and sufficient that A_0 is.*

Proof. This follows immediately from Corollary (7.2.6). \square

Proposition (7.2.9). — *Suppose the hypotheses of Proposition (7.2.7): for each integer i , let M_i be an A_i -module, and for $i \leq j$, let $v_{ij} : M_j \rightarrow M_i$ be a u_{ij} -homomorphism, such that (M_i, v_{ij}) is a projective system. In addition, suppose that M_0 is an A_0 -module of finite type and that the v_{ij} are surjective with kernel $\mathfrak{J}_j^{i+1}M_j$. Then $M = \varprojlim_i M_i$ is an A -module of finite type, and the kernel of the surjective u_n -homomorphism $v_n : M \rightarrow M_n$ is $\mathfrak{J}^{n+1}M$ (such that M_n identifies with $M/\mathfrak{J}^{n+1}M = M \otimes_A (A/\mathfrak{J}^{n+1})$).*

Proof. Let $z_h = (z_{hk})_{k \geq 0}$ be a system of s elements of M such that the z_{h0} ($1 \leq h \leq s$) forms a system of generators for M_0 ; we will show that the z_h generate the A -module M . The A -module M is

separated and complete for the filtration by the $M^{(n)}$, where $M^{(n)}$ is the set of $y = (y_k)_{k \geq 0}$ in M such that $y_k = 0$ for $k < n$; it is clear that we have $\mathfrak{J}^{(n)}M \subset M^{(n)}$ and that $M^{(n)}/M^{(n+1)} = \mathfrak{J}_n^n M_n$. We therefore have reduced to showing that the classes of the z_h modulo $M^{(0)}$ generate the graded module $\text{grad}(M)$ (by the above filtration) over the graded ring $\text{grad}(A)$ [CC, p. 18–06, lemme]; for this, we observe that it suffices to prove that the z_{h_n} ($1 \leq h \leq s$) generate the A_n -module M_n . We argue by induction on n , the proposition being evident by definition for $n = 0$; the relation $M_{n-1} = M_n/\mathfrak{J}_n^n M_n$ and the induction hypothesis show that if M'_n is the submodule of M_n generated by the z_{h_n} , we have that $M_n = M'_n + \mathfrak{J}_n^n M_n$, and as \mathfrak{J}_n is nilpotent, this implies that $M_n = M'_n$. Similarly, passing to the associated graded modules shows that the canonical map from $\mathfrak{J}^{(n)}$ to $M^{(n)}$ is surjective (thus bijection), in other words that $\mathfrak{J}^{(n)}M = \mathfrak{J}^n M$ is the kernel of $M \rightarrow M_{n-1}$. \square ErrII

Corollary (7.2.10). — *Let (N_i, w_{ij}) be a second projective system of A_i -modules satisfying the conditions of Proposition (7.2.9), and let $N = \varprojlim N_i$. There is a bijective correspondence between the projective systems (h_i) of A_i -homomorphisms $h_i : M_i \rightarrow N_i$ and the homomorphisms of A -modules $h : M \rightarrow N$ (which is necessarily continuous for the \mathfrak{J} -adic topologies).*

Proof. It is clear that if $h : M \rightarrow N$ is an A -homomorphism, then we have $h(\mathfrak{J}^n M) \subset \mathfrak{J}^n N$, hence the continuity of h ; by passing to quotients, there corresponds to h a projective system of A_i -homomorphisms $h_i : M_i \rightarrow N_i$, whose projective limit is h , hence the corollary. \square

Remark (7.2.11). — *Let A be an adic ring with an ideal of definition \mathfrak{J} such that $\mathfrak{J}/\mathfrak{J}^2$ is an A/\mathfrak{J} -module of finite type; it is clear that the $A_i = A/\mathfrak{J}^{i+1}$ satisfy the conditions of Proposition (7.2.7); as A is the projective limit of the A_i , we see that Proposition (7.2.7) gives the description of all the adic rings of the type considered (and in particular of all the adic Noetherian rings).*

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Example (7.2.12). — *Let B be a ring, \mathfrak{J} an ideal of B such that $\mathfrak{J}/\mathfrak{J}^2$ is a module of finite type over B/\mathfrak{J} (or over B , equivalently); set $A = \varprojlim_n B/\mathfrak{J}^{n+1}$; A is the separated completion of B equipped with the \mathfrak{J} -preadic topology. If $A_n = B/\mathfrak{J}^{n+1}$, then it is immediate that the A_n satisfy the conditions of Proposition (7.2.7); therefore A is an adic ring and if $\bar{\mathfrak{J}}$ is the closure in A of the canonical image of \mathfrak{J} , then $\bar{\mathfrak{J}}$ is an ideal of definition for A , $\bar{\mathfrak{J}}^n$ is the closure of the canonical image of \mathfrak{J}^n , $A/\bar{\mathfrak{J}}^n$ identifies with B/\mathfrak{J}^n and $\bar{\mathfrak{J}}/\bar{\mathfrak{J}}^2$ is isomorphic to $\mathfrak{J}/\mathfrak{J}^2$ as an $A/\bar{\mathfrak{J}}$ -module. Similarly, if N is such that $N/\mathfrak{J}N$ is a B -module of finite type, and if we set $M_i = N/\mathfrak{J}^{i+1}N$, then $M = \varprojlim M_i$ is an A -module of finite type, isomorphic to the separated completion of N for the \mathfrak{J} -preadic topology, and $\bar{\mathfrak{J}}^n M$ identifies with the closure of the canonical image of $\mathfrak{J}^n N$, and $M/\bar{\mathfrak{J}}^n M$ identifies with $N/\mathfrak{J}^n N$.*

7.3. Preadic Noetherian rings

(7.3.1). Let A be a ring, \mathfrak{J} an ideal of A , and M an A -module; we denote by $\hat{A} = \varprojlim_n A/\mathfrak{J}^n$ (resp. $\hat{M} = \varprojlim_n M/\mathfrak{J}^n M$) the separated completion of A (resp. M) for the \mathfrak{J} -preadic topology. Let $M' \xrightarrow{u} M \xrightarrow{v} M'' \rightarrow 0$ be an exact sequence of A -modules; as $M/\mathfrak{J}^n M = M \otimes_A (A/\mathfrak{J}^n)$, the sequence

$$M'/\mathfrak{J}^n M' \xrightarrow{u_n} M/\mathfrak{J}^n M \xrightarrow{v_n} M''/\mathfrak{J}^n M'' \rightarrow 0$$

is exact for each n . In addition, as $v(\mathfrak{J}^n M) = \mathfrak{J}^n v(M) = \mathfrak{J}^n M''$, $\hat{v} = \varprojlim v_n$ is surjective (Bourbaki, *Top. gén.*, Chap. IX, 2nd ed., p. 60, Cor. 2). On the other hand, if $z = (z_k)$ is an element of the kernel of \hat{v} , then for each integer k , there exists a $z'_k \in M'/\mathfrak{J}^k M'$ such that $u_k(z'_k) = z_k$; we conclude that there exists a $z' = (z'_n) \in \hat{M}'$ such that the first k components of $\hat{u}(z')$ coincide with the z ; in other words, the image of \hat{M}' under \hat{u} is dense in the kernel of \hat{v} .

If we suppose that A is Noetherian, then so is \hat{A} , according to (7.2.12), $\mathfrak{J}/\mathfrak{J}^2$ is then an A -module of finite type. In addition, we have the following theorem.

Theorem (7.3.2). — (Krull's Theorem). *Let A be a Noetherian ring, \mathfrak{J} an ideal of A , M an A -module of finite type, and M' a submodule of M ; then the induced topology on M' by the \mathfrak{J} -preadic topology of M is identical to the \mathfrak{J} -preadic topology of M' .*

This follows immediately from

Lemma (7.3.2.1). — (Artin–Rees Lemma). *Under the hypotheses of (7.3.2), there exists an integer p such that, for $n \geq p$, we have*

$$M' \cap \mathfrak{J}^n M = \mathfrak{J}^{n-p}(M' \cap \mathfrak{J}^p M).$$

For the proof, see [CC, p. 2–04].

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Corollary (7.3.3). — *Under the hypotheses of (7.3.2), the canonical map $M \otimes_A \widehat{A} \rightarrow \widehat{M}$ is bijective, and the functor $M \otimes_A \widehat{A}$ is exact in M on the category of A -modules of finite type; as a result, the separated \mathfrak{J} -adic completion \widehat{A} is a flat A -module (6.1.1).*

Proof. We first note that \widehat{M} is an exact functor in M on the category of A -modules of finite type. Indeed, let $0 \rightarrow M' \xrightarrow{u} M \xrightarrow{v} M'' \rightarrow 0$ be an exact sequence; we have seen that $\widehat{v} : \widehat{M} \rightarrow \widehat{M}''$ is surjective (7.3.1); on the other hand, if i is the canonical homomorphism $M \rightarrow \widehat{M}$, it follows from Krull’s Theorem (7.3.2) that the closure of $i(u(M'))$ in \widehat{M} identifies with the separated completion of M' for the \mathfrak{J} -preadic topology; thus \widehat{u} is injective, and according to (7.3.1), the image of \widehat{u} is equal to the kernel of \widehat{v} .

This being so, the canonical map $M \otimes_A \widehat{A} \rightarrow \widehat{M}$ is obtained by passing to the projective limit of the maps $M \otimes_A \widehat{A} \rightarrow M \otimes_A (A/\mathfrak{J}^n) = M/\mathfrak{J}^n M$. It is clear that this map is bijective when M is of the form A^p . If M is an A -module of finite type, then we have an exact sequence $A^p \rightarrow A^q \rightarrow M \rightarrow 0$, hence, by virtue of the right exactness of the functors $M \otimes_A \widehat{A}$ and \widehat{M} (in M) on the category of A -modules of finite type, we have the commutative diagram

$$\begin{array}{ccccccc} A^p \otimes_A \widehat{A} & \longrightarrow & A^q \otimes_A \widehat{A} & \longrightarrow & M \otimes_A \widehat{A} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \widehat{A}^p & \longrightarrow & \widehat{A}^q & \longrightarrow & \widehat{M} & \longrightarrow & 0, \end{array}$$

where the two rows are exact and the first two vertical arrows are isomorphisms; this immediately finishes the proof. \square

Corollary (7.3.4). — *Let A be a Noetherian ring, \mathfrak{J} an ideal of A , M and N two A -modules of finite type; we have the canonical functorial isomorphisms*

$$(M \otimes_A N)^\wedge \simeq \widehat{M} \otimes_{\widehat{A}} \widehat{N}, \quad (\text{Hom}_A(M, N))^\wedge \simeq \text{Hom}_{\widehat{A}}(\widehat{M}, \widehat{N}).$$

Proof. This follows from Corollary (7.3.3), (6.2.1), and (6.2.2). \square

Corollary (7.3.5). — *Let A be a Noetherian ring, \mathfrak{J} an ideal of A . The following conditions are equivalent:*

- (a) \mathfrak{J} is contained in the radical of A .
- (b) \widehat{A} is a faithfully flat A -module (6.4.1).
- (c) Each A -module of finite type is separated for the \mathfrak{J} -preadic topology.
- (d) Each submodule of an A -module of finite type is closed for the \mathfrak{J} -preadic topology.

Proof. As \widehat{A} is a flat A -module, the conditions (b) and (c) are equivalent, since (b) is equivalent to saying that if M is an A -module of finite type, then the canonical map $M \rightarrow \widehat{M} = M \otimes_A \widehat{A}$ is injective (6.6.1, c). It is immediate that (c) implies (d), since if N is a submodule of an A -module M of finite type, then M/N is separated for the \mathfrak{J} -preadic topology, so N is closed in M . We show that (d) implies (a): if \mathfrak{m} is a maximal ideal of A , then \mathfrak{m} is closed in A for the \mathfrak{J} -preadic topology, so $\mathfrak{m} = \bigcap_{p \geq 0} (\mathfrak{m} + \mathfrak{J}^p)$, and as $\mathfrak{m} + \mathfrak{J}^p$ is necessarily equal to A or to \mathfrak{m} , we have that $\mathfrak{m} + \mathfrak{J}^p = \mathfrak{m}$ for large enough p , hence $\mathfrak{J}^p \subset \mathfrak{m}$, and $\mathfrak{J} \subset \mathfrak{m}$ when \mathfrak{m} is prime. Finally, (a) implies (b): indeed, let P be the closure of $\{0\}$ in an A -module M of finite type, for the \mathfrak{J} -preadic topology; according to Krull’s Theorem (7.3.2), the topology induced on P by the \mathfrak{J} -preadic topology of M is the \mathfrak{J} -preadic topology of P , so $\mathfrak{J}P = P$; as P is of finite type, it follows from Nakayama’s Lemma that $P = 0$ (\mathfrak{J} being contained in the radical of A). \square

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We note that the conditions of Corollary (7.3.5) are satisfied when A is a local Noetherian ring and $\mathfrak{J} \neq A$ is any ideal of A .

Corollary (7.3.6). — *If A is a \mathfrak{J} -preadic Noetherian ring, then each A -module of finite type is separated and complete for the \mathfrak{J} -preadic topology.*

Proof. As we then have $\widehat{A} = A$, this follows immediately from Corollary (7.3.3). □

We conclude that Proposition (7.2.9) gives the description of *all* the modules of finite type over an adic Noetherian ring.

Corollary (7.3.7). — *Under the hypotheses of (7.3.2), the kernel of the canonical map $M \rightarrow \widehat{M} = M \otimes_A \widehat{A}$ is the set of the $x \in M$ killed by an element of $1 + \mathfrak{J}$.*

Proof. For each $x \in M$ in this kernel, it is necessary and sufficient that the separated completion of the submodule Ax is 0 (by Krull's Theorem (7.3.2)), in other words, that $x \in \mathfrak{J}x$. □

7.4. Quasi-finite modules over local rings

Definition (7.4.1). — Given a local ring A , with maximal ideal \mathfrak{m} , we say that an A -module M is quasi-finite (over A) if $M/\mathfrak{m}M$ is of finite rank over the residue field $k = A/\mathfrak{m}$.

When A is Noetherian, the separated completion \widehat{M} of M for the \mathfrak{m} -preadic topology is then an \widehat{A} -module of finite type; indeed, as $\mathfrak{m}/\mathfrak{m}^2$ is then an A -module of finite type, this follows from Example (7.2.12) and from the hypothesis on $M/\mathfrak{m}M$.

In particular, if we suppose that in addition A is complete and M is separated for the \mathfrak{m} -preadic topology (in other words, $\bigcap_n \mathfrak{m}^n M = 0$), then M is also an A -module of finite type: indeed, \widehat{M} is then an A -module of finite type, and as M identifies with a submodule of \widehat{M} , M is also of finite type (and is indeed identical to its completion according to Corollary (7.3.6)).

Proposition (7.4.2). — *Let A, B be two local rings, $\mathfrak{m}, \mathfrak{n}$ their maximal ideals, and suppose that B is Noetherian. Let $\phi : A \rightarrow B$ be a local homomorphism, M a B -module of finite type. If M is a quasi-finite A -module, then the \mathfrak{m} -preadic and \mathfrak{n} -preadic topologies on M are identical, thus separated.*

Proof. We note that by hypothesis $M/\mathfrak{m}M$ is of finite length as an A -module, thus also *a fortiori* as a B -module. We conclude that \mathfrak{n} is the unique prime ideal of B containing the annihilator of $M/\mathfrak{m}M$: indeed, we immediately reduce (according to (1.7.4) and (1.7.2)) to the case where $M/\mathfrak{m}M$ is simple, thus necessarily isomorphic to B/\mathfrak{n} , and our assertion is evident in this case. On the other hand, as M is a B -module of finite type, the prime ideals which contain the annihilator of $M/\mathfrak{m}M$ are those which contain $\mathfrak{m}B + \mathfrak{b}$, where we denote by \mathfrak{b} the annihilator of the B -module M (1.7.5). As B is Noetherian, we conclude ([Sam53b, p. 127, Cor. 4]) that $\mathfrak{m}B + \mathfrak{b}$ is an ideal of definition for B , in other words there exists a $k > 0$ such that $\mathfrak{n}^k \subset \mathfrak{m}B + \mathfrak{b} \subset \mathfrak{n}$; as a result, for each $h > 0$, we have

$$\mathfrak{n}^{hk} \subset (\mathfrak{m}B + \mathfrak{b})^h M = \mathfrak{m}^h M \subset \mathfrak{n}^h M,$$

which proves that the \mathfrak{m} -preadic and \mathfrak{n} -preadic topologies on M are the same; the second is separated according to Corollary (7.3.5). □

Corollary (7.4.3). — *Under the hypotheses of Proposition (7.2.4), if in addition A is Noetherian and complete for the \mathfrak{m} -preadic topology, then M is an A -module of finite type.*

Proof. Indeed, M is then separated for the \mathfrak{m} -preadic topology, and our assertion follows from the remark after Definition (7.4.1). □

(7.4.4). The most important case of Proposition (7.4.2) is when B is a quasi-finite A -module, i.e., $B/\mathfrak{m}B$ is an algebra of finite rank over $k = A/\mathfrak{m}$; furthermore, this condition can be broken down into the combination of the following:

- (i) $\mathfrak{m}B$ is an ideal of definition for B ;
- (ii) B/\mathfrak{n} is an extension of finite rank of the field A/\mathfrak{m} .

When this is so, every B -module of finite type is evidently a quasi-finite A -module.

Corollary (7.4.5). — *Under the hypotheses of Proposition (7.4.2), if \mathfrak{b} is the annihilator of the B -module M , then B/\mathfrak{b} is a quasi-finite A -module.*

Proof. Suppose $M \neq 0$ (otherwise the corollary is evident). We can consider M as a module over the local Noetherian ring B/\mathfrak{b} ; its annihilator then being 0, the proof of Proposition (7.4.2) shows that $\mathfrak{m}(B/\mathfrak{b})$ is an ideal of definition for B/\mathfrak{b} . On the other hand, M/nM is a vector space of finite rank over A/\mathfrak{m} , being a quotient of $M/\mathfrak{m}M$, which is by hypothesis of finite rank over A/\mathfrak{m} ; as $M \neq 0$, we have $M \neq nM$ by virtue of Nakayama's Lemma; as M/nM is a vector space $\neq 0$ over B/\mathfrak{n} , the fact that it is of finite rank over A/\mathfrak{m} implies that B/\mathfrak{n} is also of finite rank over A/\mathfrak{m} ; the conclusion follows from (7.4.4) applied to the ring B/\mathfrak{b} . \square

7.5. Rings of restricted formal series

(7.5.1). Let A be a topological ring, linearly topologized, separated and complete; let (\mathfrak{J}_λ) be a fundamental system of neighbourhoods of 0 in A consisting of (open) ideals, such that A canonically identifies with $\varprojlim A/\mathfrak{J}_\lambda$ (7.2.1). For each λ , let $B_\lambda = (A/\mathfrak{J}_\lambda)[T_1, \dots, T_r]$, where the T_i are indeterminates; it is clear that the B_λ form a projective system of discrete rings. We set $\varprojlim B_\lambda = A\{T_1, \dots, T_r\}$, and we will see that this topological ring is independent of the fundamental system of ideals (\mathfrak{J}_λ) considered. More precisely, let A' be the subring of the ring of formal series $A[[T_1, \dots, T_r]]$ consisting of formal series $\sum_\alpha c_\alpha T^\alpha$ (with $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbf{N}^r$) such that $\lim c_\alpha = 0$ (according to the filter by compliments of finite subsets of \mathbf{N}^r); we say that these series are the *restricted* formal series in the T_i , with coefficients in A . For each neighbourhood V of 0 in A , let V' be the set of $x = \sum_\alpha c_\alpha T^\alpha \in A'$ such that $c_\alpha \in V$ for all α . We verify immediately that the V' form a fundamental system of neighbourhoods of 0 defining on A' a *separated* ring topology; we will canonically define a *topological isomorphism* from the ring $A\{T_1, \dots, T_r\}$ to A' . For each $\alpha \in \mathbf{N}^r$ and each λ , let $\phi_{\lambda, \alpha}$ be the map from $(A/\mathfrak{J}_\lambda)[T_1, \dots, T_r]$ to A/\mathfrak{J}_λ which sends each polynomial in the first ring to coefficient of T^α in that polynomial. It is clear that the $\phi_{\lambda, \alpha}$ form a projective system of homomorphisms of A/\mathfrak{J}_λ -modules, so their projective limit is a continuous homomorphism $\phi_\alpha : A\{T_1, \dots, T_r\} \rightarrow A$; we will see that, for each $y \in A\{T_1, \dots, T_r\}$, the formal series $\sum_\alpha \phi_\alpha(y) T^\alpha$ is *restricted*. Indeed, if y_λ is the component of y in B_λ , and if we denote by H_λ the finite set of the $\alpha \in \mathbf{N}^r$ for which the coefficients of the polynomial y_λ are nonzero, then we have $\phi_{\lambda, \alpha}(y_\lambda) \in \mathfrak{J}_\lambda$ for $\mathfrak{J}_\mu \subset \mathfrak{J}_\lambda$ and $\alpha \notin H_\lambda$, and by passing to the limit, $\phi_\alpha(y) \in \mathfrak{J}_\lambda$ for $\alpha \notin H_\lambda$. We thus define a ring homomorphism $\phi : A\{T_1, \dots, T_r\} \rightarrow A'$ by setting $\phi(y) = \sum_\alpha \phi_\alpha(y) T^\alpha$, and it is immediate that ϕ is continuous. Conversely, if θ_λ is the canonical homomorphism $A \rightarrow A/\mathfrak{J}_\lambda$, then for each element $z = \sum_\alpha c_\alpha T^\alpha \in A'$ and each λ , there are only a finite number of indices α such that $\theta_\lambda(c_\alpha) \neq 0$, and as a result $\psi_\lambda(z) = \sum_\alpha \theta_\lambda(c_\alpha) T^\alpha$ is in B_λ ; the ψ_λ are continuous and form a projective system of homomorphisms whose projective limit is a continuous homomorphism $\psi : A' \rightarrow A\{T_1, \dots, T_r\}$; it remains to verify that $\phi \circ \psi$ and $\psi \circ \phi$ are the identity automorphisms, which is immediate.

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(7.5.2). We identify $A\{T_1, \dots, T_r\}$ with the ring A' of restricted formal series by means of the isomorphisms defined in (7.5.1). The canonical isomorphisms

$$((A/\mathfrak{J}_\lambda)[T_1, \dots, T_r])[T_{r+1}, \dots, T_s] \simeq (A/\mathfrak{J}_\lambda)[T_1, \dots, T_s]$$

define, by passing to the projective limit, a canonical isomorphism

$$(A\{T_1, \dots, T_r\})\{T_{r+1}, \dots, T_s\} \simeq A\{T_1, \dots, T_s\}.$$

(7.5.3). For every continuous homomorphism $u : A \rightarrow B$ from A to a linearly topologized ring B , separated and complete, and each system (b_1, \dots, b_r) of r elements of B , there exists a *unique continuous homomorphism* $\bar{u} : A\{T_1, \dots, T_r\} \rightarrow B$, such that $\bar{u}(a) = u(a)$ for all $a \in A$ and $\bar{u}(T_j) = b_j$ for $1 \leq j \leq r$. It suffices to set

$$\bar{u}\left(\sum_\alpha c_\alpha T^\alpha\right) = \sum_\alpha u(c_\alpha) b_1^{\alpha_1} \cdots b_r^{\alpha_r};$$

the verification of the fact that the family $(u(c_\alpha) b_1^{\alpha_1} \cdots b_r^{\alpha_r})$ is summable in B and that \bar{u} is continuous are immediate and left to the reader. We note that this property (for arbitrary B and b_j) *characterize* the topological ring $A\{T_1, \dots, T_r\}$ up to unique isomorphism.

Proposition (7.5.4). —

- (i) If A is an admissible ring, then so is $A' = A\{T_1, \dots, T_r\}$.

- (ii) Let A be an adic ring, \mathfrak{J} an ideal of definition for A such that $\mathfrak{J}/\mathfrak{J}^2$ is of finite type over A/\mathfrak{J} . If we set $\mathfrak{J}' = \mathfrak{J}A'$, then A' is also a \mathfrak{J}' -adic ring, and $\mathfrak{J}'/\mathfrak{J}'^2$ is of finite type over A'/\mathfrak{J}' . If in addition A is Noetherian, then so is A' .

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Proof.

- (i) If \mathfrak{J} is an ideal of A , \mathfrak{J}' the ideal of A' consisting of the $\sum_{\alpha} c_{\alpha} T^{\alpha}$ such that $c_{\alpha} \in \mathfrak{J}$ for all α , then $(\mathfrak{J}')^n \subset (\mathfrak{J}^n)'$; if \mathfrak{J} is an ideal of definition for A , then \mathfrak{J}' is also an ideal of definition for A' .
- (ii) Set $A_i = A/\mathfrak{J}^{i+1}$, and for $i \leq j$, let u_{ij} be the canonical homomorphism $A/\mathfrak{J}^{j+1} \rightarrow A/\mathfrak{J}^{i+1}$; set $A'_i = A_i[T_1, \dots, T_r]$, and let u'_{ij} be the homomorphism $A'_j \rightarrow A'_i$ ($i \leq j$) obtained by applying u_{ij} to the coefficients of the polynomials in A'_j . We will show that the projective system (A'_i, u'_{ij}) satisfies the conditions of Proposition (7.2.7); as \mathfrak{J}' is the kernel of $A' \rightarrow A'_0$, this proves the first assertion of (ii). It is clear that the u'_{ij} are surjective; the kernel \mathfrak{J}'_i of u_{0i} is the set of polynomials in $A_i[T_1, \dots, T_r]$ whose coefficients are in $\mathfrak{J}/\mathfrak{J}^{i+1}$; in particular, \mathfrak{J}'_1 is the set of polynomials in $A_1[T_1, \dots, T_r]$ whose coefficients are in $\mathfrak{J}/\mathfrak{J}^2$. As $\mathfrak{J}/\mathfrak{J}^2$ is of finite type over $A_1 = A/\mathfrak{J}^2$, we see that $\mathfrak{J}'_1/\mathfrak{J}'_1{}^2$ is a module of finite type over A'_1 (or equivalently, over $A'_0 = A'_1/\mathfrak{J}'_1$). We will now show that the kernel of u_{ij} is $\mathfrak{J}'_j{}^{i+1}$. It is evident that $\mathfrak{J}'_j{}^{i+1}$ is contained in this kernel. On the other hand, let a_1, \dots, a_m be the elements of \mathfrak{J} whose classes mod \mathfrak{J}^2 generate $\mathfrak{J}/\mathfrak{J}^2$; we verify immediately that the classes mod \mathfrak{J}^{j+1} of monomials of degree $\leq j$ in the a_k ($1 \leq k \leq m$) generate $\mathfrak{J}/\mathfrak{J}^{j+1}$, and the classes of monomials of degree $> i$ and $\leq j$ generate $\mathfrak{J}^{i+1}/\mathfrak{J}^{j+1}$; a monomial in the T_k having such an element for a coefficient is thus a product of $i+1$ elements of \mathfrak{J}'_i , which establishes our assertion. Finally, if A is Noetherian, then so is $A'/\mathfrak{J}' = (A/\mathfrak{J})[T_1, \dots, T_r]$, hence A' is Noetherian (7.2.8). □

Proposition (7.5.5). — Let A be a Noetherian \mathfrak{J} -adic ring, B an admissible topological ring, $\phi : A \rightarrow B$ a continuous homomorphism, making B an A -algebra. The following conditions are equivalent:

- (a) B is Noetherian and $\mathfrak{J}B$ -adic, and $B/\mathfrak{J}B$ is an algebra of finite type over A/\mathfrak{J} .
- (b) B is topologically A -isomorphic to $\varprojlim B_n$, where $B_n = B_m/\mathfrak{J}^{n+1}B_m$ for $m \geq n$, and B_1 is an algebra of finite type over $A_1 = A/\mathfrak{J}^2$.
- (c) B is topologically A -isomorphic to a quotient of an algebra of the form $A\{T_1, \dots, T_r\}$ by an ideal (necessarily closed according to Corollary (7.3.6) and Proposition (7.5.4, ii)).

Proof. As A is Noetherian, so is $A' = A\{T_1, \dots, T_r\}$ (7.5.4), so (c) implies that B is Noetherian; as $\mathfrak{J}' = \mathfrak{J}A'$ is an open neighbourhood of 0 in A' such that the \mathfrak{J}'^n form a fundamental system of neighbourhoods of 0, the images $\mathfrak{J}'^n B$ of the \mathfrak{J}'^n form a fundamental system of neighbourhoods of 0 in B , and as B is separated and complete, B is a $\mathfrak{J}B$ -adic ring. Finally, $B/\mathfrak{J}B$ is an algebra (over (A/\mathfrak{J}) quotient of $A'/\mathfrak{J}'A' = (A/\mathfrak{J})[T_1, \dots, T_r]$, so it is of finite type, which proves that (c) implies (a).

If B is $\mathfrak{J}B$ -adic and Noetherian, then B is isomorphic to $\varprojlim B_n$, where $B_n = B/\mathfrak{J}^{n+1}B$ (7.2.11), and $\mathfrak{J}B/\mathfrak{J}^2B$ is a module of finite type over $B/\mathfrak{J}B$. Let $(a_j)_{1 \leq j \leq s}$ be a system of generators for the $B/\mathfrak{J}B$ -module $\mathfrak{J}B/\mathfrak{J}^2B$, and let $(c_i)_{1 \leq i \leq r}$ be a system of elements of B/\mathfrak{J}^2B such that the classes mod $\mathfrak{J}B/\mathfrak{J}^2B$ form a system of generators for the A/\mathfrak{J} -algebra $B/\mathfrak{J}B$; we see immediately that the $c_i a_j$ form a system of generators for the A/\mathfrak{J}^2 -algebra B/\mathfrak{J}^2B , hence (a) implies (b). 01 | 72

It remains to prove that (b) implies (c). The hypotheses imply that B_1 is a Noetherian ring, and as $B_1 = B_2/\mathfrak{J}^2B_2$, we have $\mathfrak{J}^2B_1 = 0$, hence $\mathfrak{J}B_1 = \mathfrak{J}B_1/\mathfrak{J}^2B_1$ is a B_0 -module of finite type. The conditions of Proposition (7.2.7) are thus satisfied by the projective system (B_n) and B is a $\mathfrak{J}B$ -adic ring. Let $(c_i)_{1 \leq i \leq r}$ be a finite system of elements of B whose classes mod $\mathfrak{J}B$ generate the A/\mathfrak{J} -algebra $B/\mathfrak{J}B$, and whose linear combinations with coefficients in \mathfrak{J} are such that their classes mod \mathfrak{J}^2B generate the B_0 -module $\mathfrak{J}B/\mathfrak{J}^2B$. There exists a continuous A -homomorphism u from $A' = A\{T_1, \dots, T_r\}$ to B which reduces to ϕ on A and is such that $u(T_i) = c_i$ for $1 \leq i \leq r$ (7.5.3); if we prove that u is surjective, then (c) will be established, since from $u(A') = B$ we deduce that $u(\mathfrak{J}'A') = \mathfrak{J}'B$, in other words that u is a strict morphism of topological rings and B is this isomorphic to a quotient of A' by a closed ideal. As B is complete for the $\mathfrak{J}B$ -adic topology, it suffices ([CC, p. 18–07]) to show that the homomorphism $\text{grad}(A') \rightarrow \text{grad}(B)$, induced canonically

by u for the \mathfrak{J} -adic filtrations on A' and B , is surjective. But by definition, the homomorphisms $A'/\mathfrak{J}A' \rightarrow B/\mathfrak{J}B$ and $\mathfrak{J}A'/\mathfrak{J}^2A' \rightarrow \mathfrak{J}B/\mathfrak{J}^2B$ induced by u are surjective; by induction on n , we immediately deduce that so is $\mathfrak{J}A'/\mathfrak{J}^nA' \rightarrow \mathfrak{J}B/\mathfrak{J}^nB$, and *a fortiori* so is $\mathfrak{J}^nA'/\mathfrak{J}^{n+1}A' \rightarrow \mathfrak{J}^nB/\mathfrak{J}^{n+1}B$, which finishes the proof. \square

7.6. Completed rings of fractions

(7.6.1). Let A be a linearly topologized ring, (\mathfrak{J}_λ) a fundamental system of neighbourhoods of 0 in A consisting of ideals, S a multiplicative subset of A . Let u_λ be the canonical homomorphism $A \rightarrow A_\lambda = A/\mathfrak{J}_\lambda$, and for $\mathfrak{J}_\mu \subset \mathfrak{J}_\lambda$, let $u_{\lambda\mu}$ be the canonical homomorphism $A_\mu \rightarrow A_\lambda$. Set $S_\lambda = u_\lambda(S)$, so that $u_{\lambda\mu}(S_\mu) = S_\lambda$. The $u_{\lambda\mu}$ canonically induce surjective homomorphisms $S_\mu^{-1}A_\mu \rightarrow S_\lambda^{-1}A_\lambda$, for which these rings form a projective system; we denote by $A\{S^{-1}\}$ the projective limit of this system. This definition does not depend on the fundamental system of neighbourhoods (\mathfrak{J}_λ) chosen; indeed:

Proposition (7.6.2). — *The ring $A\{S^{-1}\}$ is topologically isomorphic to the separated completion of the ring $S^{-1}A$ for the topology which has a fundamental system of neighbourhoods of 0 consisting of the $S^{-1}\mathfrak{J}_\lambda$.*

Proof. If v_λ is the canonical homomorphism $S^{-1}A \rightarrow S_\lambda^{-1}A_\lambda$ induced by u_λ , then the kernel of v_λ is surjective, hence the proposition (7.2.1). \square

Corollary (7.6.3). — *If S' is the canonical image of S in the separated completion \widehat{A} of A , then $A\{S^{-1}\}$ canonically identifies with $\widehat{A}\{S'^{-1}\}$.*

We note that if A is separated and complete, then it is not necessarily the same for $S^{-1}A$ with the topology defined by the $S^{-1}\mathfrak{J}_\lambda$, as we see for example by taking S to be the set of the f^n ($n \geq 0$), where f is topologically nilpotent but not nilpotent: indeed, $S^{-1}A$ is not 0 and on the other hand, for each λ there exists an n such that $f^n \in \mathfrak{J}_\lambda$, so $1 = f^n/f^n \in S^{-1}\mathfrak{J}_\lambda$ and $S^{-1}\mathfrak{J}_\lambda = S^{-1}A$.

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Corollary (7.6.4). — *If, in A , 0 does not belong to S , then the ring $A\{S^{-1}\}$ is not 0.*

Proof. Indeed, 0 does not belong to $\{1\}$ in the ring $S^{-1}A$; otherwise, we would have that $1 \in S^{-1}\mathfrak{J}_\lambda$ for each open ideal \mathfrak{J}_λ of A , and it follows that $\mathfrak{J}_\lambda \cap S \neq \emptyset$ for all λ , contradicting the hypothesis. \square

(7.6.5). We say that $A\{S^{-1}\}$ is the *completed ring of fractions* of A with denominators in S . With the above notation, it is clear that the inverse image of $S^{-1}\mathfrak{J}_\lambda$ in A contains \mathfrak{J}_λ , hence the canonical map $A \rightarrow S^{-1}A$ is continuous, and if we compose it with the canonical map $S^{-1}A \rightarrow A\{S^{-1}\}$, we obtain a canonical continuous homomorphism $A \rightarrow A\{S^{-1}\}$, the projective limit of the homomorphisms $A \rightarrow S_\lambda^{-1}A_\lambda$.

(7.6.6). The couple consisting of $A\{S^{-1}\}$ and the canonical homomorphism $A \rightarrow A\{S^{-1}\}$ are characterized by the following *universal property*: every continuous homomorphism u from A to a linearly topologized ring B , separated and complete, such that $u(S)$ consists of the invertible elements of B , uniquely factorizes as $A \rightarrow A\{S^{-1}\} \xrightarrow{u'} B$, where u' is continuous. Indeed, u uniquely factorizes as $A \rightarrow S^{-1}A \xrightarrow{v'} B$; as for each open ideal \mathfrak{K} of B we have that $u^{-1}(\mathfrak{K})$ contains a \mathfrak{J}_λ , $v'^{-1}(\mathfrak{K})$ necessarily contains $S^{-1}\mathfrak{J}_\lambda$, so v' is continuous; since B is separated and complete, v' uniquely factorizes as $S^{-1}A \rightarrow A\{S^{-1}\} \xrightarrow{u'} B$, where u' is continuous; hence our assertion.

(7.6.7). Let B be a second linearly topologized ring, T a multiplicative subset of B , $\phi : A \rightarrow B$ a continuous homomorphism such that $\phi(S) \subset T$. According to the above, the continuous homomorphism $A \xrightarrow{\phi} B \rightarrow B\{T^{-1}\}$ uniquely factorizes as $A \rightarrow A\{S^{-1}\} \xrightarrow{\phi'} B\{T^{-1}\}$, where ϕ' is continuous. In particular, if $B = A$ and if ϕ is the identity, we see that for $S \subset T$ we have a continuous homomorphism $\rho^{T,S} : A\{S^{-1}\} \rightarrow A\{T^{-1}\}$ obtained by passing to the separated completion from $S^{-1}A \rightarrow T^{-1}A$; if U is a third multiplicative subset of A such that $S \subset T \subset U$, then we have $\rho^{U,S} = \rho^{U,T} \circ \rho^{T,S}$.

(7.6.8). Let S_1, S_2 be two multiplicative subsets of A , and let S'_2 be the canonical image of S_2 in $A\{S_1^{-1}\}$; we then have a canonical topological isomorphism $A\{(S_1S_2)^{-1}\} \simeq A\{S_1^{-1}\}\{S'_2\}^{-1}$, as we see from the canonical isomorphism $(S_1S_2)^{-1}A \simeq S_2'^{-1}(S_1^{-1}A)$ (where S_2'' is the canonical image of S_2 in $S_1^{-1}A$), which is bicontinuous.

(7.6.9). Let \mathfrak{a} be an *open* ideal of A ; we can assume that $\mathfrak{J}_\lambda \subset \mathfrak{a}$ for all λ , and as a result $S^{-1}\mathfrak{J}_\lambda \subset S^{-1}\mathfrak{a}$ in the ring $S^{-1}A$, in other words, $S^{-1}\mathfrak{a}$ is an *open* ideal of $S^{-1}A$; we denote by $\mathfrak{a}\{S^{-1}\}$ its separated completion, equal to $\varprojlim(S^{-1}\mathfrak{a}/S^{-1}\mathfrak{J}_\lambda)$, which is an *open* ideal of $A\{S^{-1}\}$, isomorphic to the closure of the canonical image of $S^{-1}\mathfrak{a}$. In addition, *the discrete ring $A\{S^{-1}\}/\mathfrak{a}\{S^{-1}\}$ is canonically isomorphic to $S^{-1}A/S^{-1}\mathfrak{a} = S^{-1}(A/\mathfrak{a})$. Conversely, if \mathfrak{a}' is an open ideal of $A\{S^{-1}\}$, then \mathfrak{a}' contains an ideal of the form $\mathfrak{J}_\lambda\{S^{-1}\}$ which is the inverse image of an ideal of $S^{-1}A/S^{-1}\mathfrak{J}_\lambda$, which is necessarily (1.2.6) of the form $S^{-1}\mathfrak{a}$, where $\mathfrak{a} \supset \mathfrak{J}_\lambda$. We conclude that $\mathfrak{a}' = \mathfrak{a}\{S^{-1}\}$. In particular (1.2.6):*

Proposition (7.6.10). — *The map $\mathfrak{p} \mapsto \mathfrak{p}\{S^{-1}\}$ is an increasing bijection from the set of open prime ideals \mathfrak{p} of A such that $\mathfrak{p} \cap S = \emptyset$ to the set of open prime ideals of $A\{S^{-1}\}$; in addition, the field of fractions of $A\{S^{-1}\}/\mathfrak{p}\{S^{-1}\}$ is canonically isomorphic to that of A/\mathfrak{p} .*

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Proposition (7.6.11). —

- (i) *If A is an admissible ring, then so is $A' = A\{S^{-1}\}$, and for every ideal of definition \mathfrak{J} for A , $\mathfrak{J}' = \mathfrak{J}\{S^{-1}\}$ is an ideal of definition for A' .*
- (ii) *Let A be an adic ring, \mathfrak{J} an ideal of definition for A such that $\mathfrak{J}/\mathfrak{J}^2$ is of finite type over A/\mathfrak{J} ; then A' is a \mathfrak{J}' -adic ring and $\mathfrak{J}'/\mathfrak{J}'^2$ is of finite type over A'/\mathfrak{J}' . If in addition A is Noetherian, then so is A' .*

Proof.

- (i) If \mathfrak{J} is an ideal of definition for A , then it is clear that $S^{-1}\mathfrak{J}$ is an ideal of definition for the topological ring $S^{-1}A$, since we have $(S^{-1}\mathfrak{J})^n = S^{-1}\mathfrak{J}^n$. Let A'' be the separated ring associated to $S^{-1}A$, \mathfrak{J}'' the image of $S^{-1}\mathfrak{J}$ in A'' ; the image of $S^{-1}\mathfrak{J}^n$ is \mathfrak{J}''^n , so \mathfrak{J}''^n tends to 0 in A'' ; as \mathfrak{J}' is the closure of \mathfrak{J}'' in A' , \mathfrak{J}'^n is contained in the closure of \mathfrak{J}''^n , hence tends to 0 in A' .
- (ii) Set $A_i = A/\mathfrak{J}^{i+1}$, and for $i \leq j$, let u_{ij} be the canonical homomorphism $A/\mathfrak{J}^{j+1} \rightarrow A/\mathfrak{J}^{i+1}$; let S_i be the canonical image of S in A_i , and set $A'_i = S_i^{-1}A_i$; finally, let $u'_{ij} : A'_j \rightarrow A'_i$ be the homomorphism canonically induced by u_{ij} . We show that the projective system (A'_i, u'_{ij}) satisfies the conditions of Proposition (7.2.7): it is clear that the u'_{ij} are surjective; on the other hand, the kernel of u'_{ij} is $S_j^{-1}(\mathfrak{J}^{i+1}/\mathfrak{J}^{j+1})$ (1.3.2), equal to $\mathfrak{J}'_j{}^{i+1}$, where $\mathfrak{J}'_j = S_j^{-1}(\mathfrak{J}/\mathfrak{J}^{j+1})$; finally, $\mathfrak{J}'_1/\mathfrak{J}'_1{}^2 = S_1^{-1}(\mathfrak{J}/\mathfrak{J}^2)$, and as $\mathfrak{J}/\mathfrak{J}^2$ is of finite type over A/\mathfrak{J}^2 , $\mathfrak{J}'_1/\mathfrak{J}'_1{}^2$ is of finite type over A'_1 . Finally, if A is Noetherian, then so is $A'_0 = S_0^{-1}(A/\mathfrak{J})$, which finishes the proof (7.2.8). □

Corollary (7.6.12). — *Under the hypotheses of Proposition (7.6.11, ii), we have $(\mathfrak{J}\{S^{-1}\})^n = \mathfrak{J}^n\{S^{-1}\}$.*

Proof. This follows from Proposition (7.2.7) and the proof of Proposition (7.6.11). □

Proposition (7.6.13). — *Let A be an adic Noetherian ring, S a multiplicative subset of A ; then $A\{S^{-1}\}$ is a flat A -module.*

Proof. If \mathfrak{J} is an ideal of definition for A , then $A\{S^{-1}\}$ is the separated completion of the Noetherian ring $S^{-1}A$ equipped with the $S^{-1}\mathfrak{J}$ -preadic topology; as a result (7.3.3) $A\{S^{-1}\}$ is a flat $S^{-1}A$ -module; as $S^{-1}A$ is a flat A -module (6.3.1), the proposition follows from the transitivity of flatness (6.2.1). □

Corollary (7.6.14). — *Under the hypotheses of Proposition (7.6.13), let $S' \subset S$ be a second multiplicative subset of A ; then $A\{S^{-1}\}$ is a flat $A\{S'^{-1}\}$ -module.*

Proof. By (7.6.8), $A\{S^{-1}\}$ canonically identifies with $A\{S'^{-1}\}\{S_0^{-1}\}$, where S_0 is the canonical image of S in $A\{S'^{-1}\}$, and $A\{S'^{-1}\}$ is Noetherian (7.6.11). □

(7.6.15). For each element f of a linearly topologized ring A , we denote by $A_{\{f\}}$ the completed ring of fractions $A\{S_f^{-1}\}$, where S_f is the multiplicative set of the f^n ($n \geq 0$); for each open ideal \mathfrak{a} of A , we write $\mathfrak{a}_{\{f\}}$ for $\mathfrak{a}\{S_f^{-1}\}$. If g is a second element of A , then we have a canonical continuous homomorphism $A_{\{f\}} \rightarrow A_{\{fg\}}$ (7.6.7). When f varies over a multiplicative subset S of A , the $A_{\{f\}}$

form a filtered inductive system with the above homomorphisms; we set $A_{\{S\}} = \varinjlim_{f \in S} A_{\{f\}}$. For every $f \in S$, we have a homomorphism $A_{\{f\}} \rightarrow A\{S^{-1}\}$ (7.6.7), and these homomorphisms form an inductive system; by passing to the inductive limit, they thus define a canonical homomorphism $A_{\{S\}} \rightarrow A\{S^{-1}\}$.

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Proposition (7.6.16). — *If A is a Noetherian ring, then $A\{S^{-1}\}$ is a flat module over $A_{\{S\}}$.*

Proof. By (7.6.14), $A\{S^{-1}\}$ is flat for each of the rings $A_{\{f\}}$ for $f \in S$, and the conclusion follows from (6.2.3). \square

Proposition (7.6.17). — *Let \mathfrak{p} be an open prime ideal in an admissible ring A , and let $S = A - \mathfrak{p}$. Then the rings $A\{S^{-1}\}$ and $A_{\{S\}}$ are local rings, the canonical homomorphism $A_{\{S\}} \rightarrow A\{S^{-1}\}$ is local, and the residue fields of $A_{\{S\}}$ and $A\{S^{-1}\}$ are canonically isomorphic to the field of fractions of A/\mathfrak{p} .*

Proof. Let $\mathfrak{J} \subset \mathfrak{p}$ be an ideal of definition for A ; we have $S^{-1}\mathfrak{J} \subset S^{-1}\mathfrak{p} = \mathfrak{p}A_{\mathfrak{p}}$, so $A_{\mathfrak{p}}/S^{-1}\mathfrak{J}$ is a local ring; we conclude from Corollary (7.1.12), (7.6.9), and Proposition (7.6.11, i) that $A\{S^{-1}\}$ is a local ring. Set $\mathfrak{m} = \varinjlim_{f \in S} \mathfrak{p}_{\{f\}}$, which is an ideal in $A_{\{S\}}$; we will see that each element in $A_{\{S\}}$ not in \mathfrak{m} is invertible. Indeed, such an element is the image in $A_{\{S\}}$ of an element $z \in A_{\{f\}}$ not in $\mathfrak{p}_{\{f\}}$, for an $f \in S$; its canonical image z_0 in $A_{\{f\}}/\mathfrak{J}_{\{f\}} = S_f^{-1}(A/\mathfrak{J})$ therefore is not in $S_f^{-1}(\mathfrak{p}/\mathfrak{J})$ (7.6.9), which means that $z_0 = \bar{x}/\bar{f}^k$, where $x \notin \mathfrak{p}$ and \bar{x}, \bar{f} are the classes of $x, f \bmod \mathfrak{J}$. As $x \in S$, we have $g = xf \in S$, and in $S_g^{-1}A$, the canonical image $y_0 = x^{k+1}/g^k$ of $x/f^k \in S_f^{-1}A$ admits an inverse $x^{k-1}f^{2k}/g^k$. This implies *a fortiori* that the image of y_0 in $S_g^{-1}A/S_g^{-1}\mathfrak{J}$ is invertible, so ((7.6.9) and Corollary (7.1.12)) the canonical image y of z in $A_{\{g\}}$ is invertible; the image of z in $A_{\{S\}}$ (equal to that of y) is as a result invertible. We thus see that $A_{\{S\}}$ a local ring with maximal ideal \mathfrak{m} ; in addition, the image of $\mathfrak{p}_{\{f\}}$ in $A\{S^{-1}\}$ is contained in the maximal ideal $\mathfrak{p}\{S^{-1}\}$ of this ring; *a fortiori*, the image of \mathfrak{m} in $A\{S^{-1}\}$ is contained in $\mathfrak{p}\{S^{-1}\}$, so the canonical homomorphism $A_{\{S\}} \rightarrow A\{S^{-1}\}$ is local. Finally, as each element of $A\{S^{-1}\}/\mathfrak{p}\{S^{-1}\}$ is the image of an element in the ring $S_f^{-1}A$ for a suitable $f \in S$, the homomorphism $A_{\{S\}} \rightarrow A\{S^{-1}\}/\mathfrak{p}\{S^{-1}\}$ is surjective, and gives an isomorphism of the residue fields by passing to quotients. \square

Corollary (7.6.18). — *Under the hypotheses of Proposition (7.6.17), if we suppose also that A is an adic Noetherian ring, then the local rings $A\{S^{-1}\}$ and $A_{\{S\}}$ are Noetherian, and $A\{S^{-1}\}$ is a faithfully flat $A_{\{S\}}$ -module.*

Proof. We know from before (7.6.11, ii) that $A\{S^{-1}\}$ is Noetherian and $A_{\{S\}}$ -flat (7.6.16); as the homomorphism $A_{\{S\}} \rightarrow A\{S^{-1}\}$ is local, we conclude that $A\{S^{-1}\}$ is a faithfully flat $A_{\{S\}}$ -module (6.6.2), and as a result that $A_{\{S\}}$ is Noetherian (6.5.2). \square

7.7. Completed tensor products

(7.7.1). Let A be a linearly topologized ring, M, N two linearly topologized A -modules. Let \mathfrak{J}, V, W be open neighbourhoods of 0 in A, M, N respectively, which are A -modules, and such that $\mathfrak{J}M \subset V, \mathfrak{J}N \subset W$, so that M/V and N/W can be considered as A/\mathfrak{J} -modules. When \mathfrak{J}, V, W vary over the systems of open neighbourhoods satisfying these properties, it is immediate that the modules $(M/V) \otimes_{A/\mathfrak{J}} (N/W)$ form a projective system of modules over the projective system of rings A/\mathfrak{J} ; by passing to the projective limit, we thus obtain a module over the separated completion \widehat{A} of A , which we call the *completed tensor product* of M and N and denote by $(M \otimes_A N)^\wedge$. If we have that M/V is canonically isomorphic to \widehat{M}/\overline{V} , where \widehat{M} is the separated completion of M and \overline{V} the closure in \widehat{M} of the image of V , then we see that the completed tensor product $(M \otimes_A N)^\wedge$ canonically identifies with $(\widehat{M} \otimes_{\widehat{A}} \widehat{N})^\wedge$, which we denote by $\widehat{M} \widehat{\otimes}_{\widehat{A}} \widehat{N}$.

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(7.7.2). With the above notation, the tensor products $(M/V) \otimes_A (N/W)$ and $(M/V) \otimes_{A/\mathfrak{J}} (N/W)$ identify canonically; they identify with $(M \otimes_A N)/(\text{Im}(V \otimes_A N) + \text{Im}(M \otimes_A W))$. We conclude that $(M \otimes_A N)^\wedge$ is the separated completion of the A -module $M \otimes_A N$, equipped with the topology for which the submodules

$$\text{Im}(V \otimes_A N) + \text{Im}(M \otimes_A W)$$

form a fundamental system of neighbourhoods of 0 (V and W varying over the set of open submodules of M and N respectively); we say for brevity that this topology is the *tensor product* of the given topologies on M and N .

(7.7.3). Let M', N' be two linearly topologized A -modules, $u : M \rightarrow M', v : N \rightarrow N'$ two continuous homomorphisms; it is immediate that $u \otimes v$ is continuous for the tensor product topologies on $M \otimes_A N$ and $M' \otimes_A N'$ respectively; by passing to the separated completions, we obtain a continuous homomorphism $(M \otimes_A N)^\wedge \rightarrow (M' \otimes_A N')^\wedge$, which we denote by $u \widehat{\otimes} v$; $(M \otimes_A N)^\wedge$ is thus a *bifunctor* in M and N on the category of linearly topologized A -modules.

(7.7.4). We similarly define the completed tensor product of any finite number of linearly topologized A -modules; it is immediate that this product has the usual properties of associativity and commutativity.

(7.7.5). If B, C are two linearly topologized A -algebras, then the tensor product topology on $B \otimes_A C$ has for a fundamental system of neighbourhoods of 0 the *ideals* $\text{Im}(\mathfrak{K} \otimes_A C) + \text{Im}(B \otimes_A \mathfrak{L})$ of the algebra $B \otimes_A C$, \mathfrak{K} (resp. \mathfrak{L}) varying over the set of open ideals of B (resp. C). As a result, $(B \otimes_A C)^\wedge$ is equipped with the structure of a *topological \widehat{A} -algebra*, the projective limit of the projective system of A/\mathfrak{J} -algebras $(B/\mathfrak{K}) \otimes_{A/\mathfrak{J}} (C/\mathfrak{L})$ (\mathfrak{J} the open ideal of A such that $\mathfrak{J}B \subset \mathfrak{K}, \mathfrak{J}C \subset \mathfrak{L}$; it always exists). We say that this algebra is the *completed tensor product* of the algebras B and C .

(7.7.6). The A -algebra homomorphisms $b \mapsto b \otimes 1, c \mapsto 1 \otimes c$ from B and C to $B \otimes_A C$ are continuous when we equip the latter algebra with the tensor product topology; by composing with the canonical homomorphism from $B \otimes_A C$ to its separated completion, they give *canonical homomorphisms* $\rho : B \rightarrow (B \otimes_A C)^\wedge, \sigma : C \rightarrow (B \otimes_A C)^\wedge$. The algebra $(B \otimes_A C)^\wedge$ and the homomorphisms ρ and σ have in addition the following *universal property*: for every linearly topologized A -algebra D , separated and complete, and each pair of continuous A -homomorphisms $u : B \rightarrow D, v : C \rightarrow D$, there exists a unique continuous A -homomorphism $w : (B \otimes_A C)^\wedge \rightarrow D$ such that $u = w \circ \rho$ and $v = w \circ \sigma$. Indeed, there already exists a unique A -homomorphism $w_0 : B \otimes_A C \rightarrow D$ such that $u(b) = w_0(b \otimes 1)$ and $v(c) = w_0(1 \otimes c)$, and it remains to prove that w_0 is *continuous*, since it then gives a continuous homomorphism w by passing to the separated completion. If \mathfrak{M} is an open ideal of D , then there exists by hypothesis open ideals $\mathfrak{K} \subset B, \mathfrak{L} \subset C$ such that $u(\mathfrak{K}) \subset \mathfrak{M}, v(\mathfrak{L}) \subset \mathfrak{M}$; the image under w_0 of $\text{Im}(\mathfrak{K} \otimes_A C) + \text{Im}(B \otimes_A \mathfrak{L})$ is again contained in \mathfrak{M} , hence our assertion.

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Proposition (7.7.7). — *If B and C are two preadmissible A -algebras, then $(B \otimes_A C)^\wedge$ is admissible, and if \mathfrak{K} (resp. \mathfrak{L}) is an ideal of definition for B (resp. C), then the closure in $(B \otimes_A C)^\wedge$ of the canonical image of $\mathfrak{H} = \text{Im}(\mathfrak{K} \otimes_A C) + \text{Im}(B \otimes_A \mathfrak{L})$ is an ideal of definition.*

Proof. It suffices to show that \mathfrak{H}^n tends to 0 for the tensor product topology, which follows immediately from the inclusion

$$\mathfrak{H}^{2n} \subset \text{Im}(\mathfrak{K}^n \otimes_A C) + \text{Im}(B \otimes_A \mathfrak{L}^n).$$

□

Proposition (7.7.8). — *Let A be a preadic ring, \mathfrak{J} an ideal of definition for A , M an A -module of finite type, equipped with the \mathfrak{J} -preadic topology. For every topological adic Noetherian A -algebra B , $B \otimes_A M$ identifies with the completed tensor product $(B \otimes_A M)^\wedge$.*

Proof. If \mathfrak{K} is an ideal of definition for B , there exists by hypothesis an integer m such that $\mathfrak{J}^m B \subset \mathfrak{K}$, so $\text{Im}(B \otimes_A \mathfrak{J}^{nm} M) = \text{Im}(\mathfrak{J}^{nm} B \otimes_A M) \subset \text{Im}(\mathfrak{K}^n B \otimes_A M) = \mathfrak{K}^n (B \otimes_A M)$; we conclude that over $B \otimes_A M$, the tensor products of the topologies of B and M is the \mathfrak{K} -preadic topology. As $B \otimes_A M$ is a B -module of finite type, the proposition follows from Corollary (7.3.6). □

7.8. Topologies on modules of homomorphisms

(7.8.1). Let A be a Noetherian \mathfrak{J} -adic ring, M and N two A -modules of finite type, equipped with the \mathfrak{J} -preadic topology; we know (7.3.6) that they are separated and complete; in addition, every A -homomorphism $M \rightarrow N$ is automatically continuous, and the A -module $\text{Hom}_A(M, N)$ is of finite type. For every integer $i \geq 0$, set $A_i = A/\mathfrak{J}^{i+1}$, $M_i = M/\mathfrak{J}^{i+1}M$, $N_i = N/\mathfrak{J}^{i+1}N$; for $i \leq j$, every homomorphism $u_j : M_j \rightarrow N_j$ maps $\mathfrak{J}^{i+1}M_j$ to $\mathfrak{J}^{i+1}N_j$, thus giving by passage to quotients a homomorphism $u_i : M_i \rightarrow N_i$, which defines a canonical homomorphism $\text{Hom}_{A_j}(M_j, N_j) \rightarrow \text{Hom}_{A_i}(M_i, N_i)$; in addition, the $\text{Hom}_{A_i}(M_i, N_i)$ form a projective system for these homomorphisms, and it follows from Corollary (7.2.10) that there is a canonical isomorphism $\phi : \text{Hom}_A(M, N) \rightarrow \varprojlim_i \text{Hom}_{A_i}(M_i, N_i)$. In addition:

Proposition (7.8.2). — *If M and N are modules of finite type over a \mathfrak{J} -adic Noetherian ring A , then the submodules $\text{Hom}_A(M, \mathfrak{J}^{i+1}N)$ form a fundamental system of neighbourhoods of 0 in $\text{Hom}_A(M, N)$ for the \mathfrak{J} -adic topology, and the canonical isomorphism $\phi : \text{Hom}_A(M, N) \rightarrow \varprojlim_i \text{Hom}_{A_i}(M_i, N_i)$ is a topological isomorphism.*

Proof. We can consider M as the quotient of a free A -module L of finite type, and as a result identify $\text{Hom}_A(M, N)$ as a submodule of $\text{Hom}_A(L, N)$; in this identification, $\text{Hom}_A(M, \mathfrak{J}^{i+1}N)$ is the intersection of $\text{Hom}_A(M, N)$ and $\text{Hom}_A(L, \mathfrak{J}^{i+1}N)$; as the induced topology on $\text{Hom}_A(M, N)$ by the \mathfrak{J} -adic topology of $\text{Hom}_A(L, N)$ is the \mathfrak{J} -adic (7.3.2), we have reduced to proving the first assertion for $M = L = A^m$; but then $\text{Hom}_A(L, N) = N^m$, $\text{Hom}_A(L, \mathfrak{J}^{i+1}N) = (\mathfrak{J}^{i+1}N)^m = \mathfrak{J}^{i+1}N^m$ and the result is evident. To establish the second assertion, we note that the image of $\text{Hom}_A(M, \mathfrak{J}^{i+1}N)$ in $\text{Hom}_{A_j}(M_j, N_j)$ is zero for $j \leq i$, hence ϕ is continuous; conversely, the inverse image in $\text{Hom}_A(M, N)$ of 0 of $\text{Hom}_{A_i}(M_i, N_i)$ is $\text{Hom}_A(M, \mathfrak{J}^{i+1}N)$, so ϕ is bicontinuous. \square

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If we only suppose that A is a Noetherian \mathfrak{J} -preadic ring, M and N two A -modules of finite type, separated for the \mathfrak{J} -preadic topology, then the following proof shows that the first assertion of Proposition (7.8.2) remains valid, and that ϕ is a topological isomorphism from $\text{Hom}_A(M, N)$ to a submodule of $\varprojlim_i \text{Hom}_{A_i}(M_i, N_i)$.

Proposition (7.8.3). — *Under the hypotheses of Proposition (7.8.2), the set of injective (resp. surjective, bijective) homomorphisms from M to N is an open subset of $\text{Hom}_A(M, N)$.*

Proof. According to Corollaries (7.3.5) and (7.1.14), for u to be injective, it is necessary and sufficient that the corresponding homomorphism $u_0 : M/\mathfrak{J}M \rightarrow N/\mathfrak{J}N$ is, and the set of surjective homomorphisms from M to N is thus the inverse image under the continuous map $\text{Hom}_A(M, N) \rightarrow \text{Hom}_{A_0}(M_0, N_0)$ of a subset of a discrete space. We now show that the set of injective homomorphisms is open; let v be such a homomorphism and set $M' = v(M)$; by the Artin–Rees Lemma (7.3.2.1), there exists an integer $k \geq 0$ such that $M' \cap \mathfrak{J}^{m+k}N \subset \mathfrak{J}^m M'$ for all $m > 0$; we will see that for all $w \in \mathfrak{J}^{k+1} \text{Hom}_A(M, N)$, $u = v + w$ is injective, which will finish the proof. Indeed, let $x \in M$ be such that $u(x) = 0$; we prove that for every $i \geq 0$ the relation $x \in \mathfrak{J}^i M$ implies that $x \in \mathfrak{J}^{i+1}M$; this follows from $x \in \bigcap_{i \geq 0} \mathfrak{J}^i M = (0)$. Indeed, we then have $w(x) \in \mathfrak{J}^{i+k+1}N$, and as a result $w(x) = -v(x) \in M'$, so $v(x) \in M' \cap \mathfrak{J}^{i+k+1}N \subset \mathfrak{J}^{i+1}M'$, and as v is an isomorphism from M to M' , $x \in \mathfrak{J}^{i+1}M$; q.e.d. \square

§8. REPRESENTABLE FUNCTORS

8.1. Representable functors

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(8.1.1). We denote by Set the category of sets. Let \mathcal{C} be a category; for two objects X, Y of \mathcal{C} , we set $h_X(Y) = \text{Hom}(Y, X)$; for each morphism $u : Y \rightarrow Y'$ in \mathcal{C} , we denote by $h_X(u)$ the map $v \mapsto vu$ from $\text{Hom}(Y', X)$ to $\text{Hom}(Y, X)$. It is immediate that with these definitions, $h_X : \mathcal{C} \rightarrow \text{Set}$ is a contravariant functor, i.e., an object of the category $\text{Hom}(\mathcal{C}^{\text{op}}, \text{Set})$, of covariant functors from the category \mathcal{C}^{op} (the dual of the category \mathcal{C}) to the category Set (T, 1.7, (d) and [Car]).

(8.1.2). Now let $w : X \rightarrow X'$ be a morphism in \mathcal{C} ; for each $Y \in \mathcal{C}$ and each $v \in \text{Hom}(Y, X) = h_X(Y)$, we have $wv \in \text{Hom}(Y, X') = h_{X'}(Y)$; we denote by $h_w(Y)$ the map $v \mapsto wv$ from $h_X(Y)$ to $h_{X'}(Y)$. It is immediate that for each morphism $u : Y \rightarrow Y'$ in \mathcal{C} , the diagram

$$\begin{array}{ccc} h_X(Y') & \xrightarrow{h_X(u)} & h_X(Y) \\ h_w(Y') \downarrow & & \downarrow h_w(Y) \\ h_{X'}(Y') & \xrightarrow{h_{X'}(u)} & h_{X'}(Y) \end{array}$$

is commutative; in other words, h_w is a *natural transformation* (or *functorial morphism*) $h_X \rightarrow h_{X'}$ (T, 1.2), also a morphism in the category $\text{Hom}(\mathcal{C}^{\text{op}}, \text{Set})$ (T, 1.7, (d)). The definitions of h_X and of h_w therefore constitute the definition of a *canonical covariant functor*

$$(8.1.2.1) \quad h : \mathcal{C} \longrightarrow \text{Hom}(\mathcal{C}^{\text{op}}, \text{Set}), \quad X \longmapsto h_X.$$

(8.1.3). Let X be an object in \mathcal{C} , F a contravariant functor from \mathcal{C} to Set (an object of $\text{Hom}(\mathcal{C}^{\text{op}}, \text{Set})$). Let $g : h_X \rightarrow F$ be a *natural transformation*: for all $Y \in \mathcal{C}$, $g(Y)$ is thus a map $h_X(Y) \rightarrow F(Y)$ such that for each morphism $u : Y \rightarrow Y'$ in \mathcal{C} , the diagram

$$(8.1.3.1) \quad \begin{array}{ccc} h_X(Y') & \xrightarrow{h_X(u)} & h_X(Y) \\ g(Y') \downarrow & & \downarrow g(Y) \\ F(Y') & \xrightarrow{F(u)} & F(Y) \end{array}$$

is commutative. In particular, we have a map $g(X) : h_X(X) = \text{Hom}(X, X) \rightarrow F(X)$, hence an element

$$(8.1.3.2) \quad \alpha(g) = (g(X))(1_X) \in F(X)$$

and as a result a canonical map

$$(8.1.3.3) \quad \alpha : \text{Hom}(h_X, F) \longrightarrow F(X).$$

Conversely, consider an element $\zeta \in F(X)$; for each morphism $v : Y \rightarrow X$ in \mathcal{C} , $F(v)$ is a map $F(X) \rightarrow F(Y)$; consider the map

$$(8.1.3.4) \quad v \longmapsto (F(v))(\zeta)$$

from $h_X(Y)$ to $F(Y)$; if we denote by $(\beta(\zeta))(Y)$ this map,

$$(8.1.3.5) \quad \beta(\zeta) : h_X \longrightarrow F$$

is a *natural transformation*, since for each morphism $u : Y \rightarrow Y'$ in \mathcal{C} we have $(F(vu))(\zeta) = (F(v) \circ F(u))(\zeta)$, which makes (8.1.3.1) commutative for $g = \beta(\zeta)$. We have thus defined a canonical map

$$(8.1.3.6) \quad \beta : F(X) \longrightarrow \text{Hom}(h_X, F).$$

Proposition (8.1.4). — *The maps α and β are the inverse bijections of each other.*

Proof. We calculate $\alpha(\beta(\zeta))$ for $\zeta \in F(X)$; for each $Y \in \mathcal{C}$, $(\beta(\zeta))(Y)$ is a map $g_1(Y) : v \mapsto (F(v))(\zeta)$ from $h_X(Y)$ to $F(Y)$. We thus have

$$\alpha(\beta(\zeta)) = (g_1(X))(1_X) = (F(1_X))(\zeta) = 1_{F(X)}(\zeta) = \zeta.$$

We now calculate $\beta(\alpha(g))$ for $g \in \text{Hom}(h_X, F)$; for each $Y \in \mathcal{C}$, $(\beta(\alpha(g)))(Y)$ is the map $v \mapsto (F(v))((g(X))(1_X))$; according to the commutativity of (8.1.3.1), this map is none other than $v \mapsto (g(Y))((h_X(v))(1_X)) = (g(Y))(v)$ by definition of $h_X(v)$, in other words, it is equal to $g(Y)$, which finishes the proof. \square

(8.1.5). Recall that a *subcategory* \mathcal{C}' of a category \mathcal{C} is defined by the condition that its objects are objects of \mathcal{C} , and that if X', Y' are two objects of \mathcal{C}' , then the set $\text{Hom}_{\mathcal{C}'}(X', Y')$ of morphisms $X' \rightarrow Y'$ in \mathcal{C}' is a subset of the set $\text{Hom}_{\mathcal{C}}(X', Y')$ of morphisms $X' \rightarrow Y'$ in \mathcal{C} , the canonical map of “composition of morphisms”

$$\text{Hom}_{\mathcal{C}'}(X', Y') \times \text{Hom}_{\mathcal{C}'}(Y', Z') \longrightarrow \text{Hom}_{\mathcal{C}'}(X', Z')$$

being the restriction of the canonical map

$$\mathrm{Hom}_{\mathcal{C}}(X', Y') \times \mathrm{Hom}_{\mathcal{C}}(Y', Z') \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X', Z').$$

We say that \mathcal{C}' is a *full* subcategory of \mathcal{C} if $\mathrm{Hom}_{\mathcal{C}'}(X', Y') = \mathrm{Hom}_{\mathcal{C}}(X', Y')$ for every pair of objects in \mathcal{C}' . The subcategory \mathcal{C}'' of \mathcal{C} consisting of the objects of \mathcal{C} isomorphic to objects of \mathcal{C}' is then again a full subcategory of \mathcal{C} , *equivalent* (T, 1.2) to \mathcal{C}' as we verify easily.

A covariant functor $F : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ is called *fully faithful* if for every pair of objects X_1, Y_1 of \mathcal{C}_1 , the map $u \mapsto F(u)$ from $\mathrm{Hom}(X_1, Y_1)$ to $\mathrm{Hom}(F(X_1), F(Y_1))$ is *bijective*; this implies that the subcategory $F(\mathcal{C}_1)$ of \mathcal{C}_2 is *full*. In addition, if two objects X_1, X'_1 have the same image X_2 , then there exists a unique isomorphism $u : X_1 \rightarrow X'_1$ such that $F(u) = 1_{X_2}$. For each object X_2 of $F(\mathcal{C}_1)$, let $G(X_2)$ be one of the objects X_1 of \mathcal{C}_1 such that $F(X_1) = X_2$ (G is defined by means of the axiom of choice); for each morphism $v : X_2 \rightarrow Y_2$ in $F(\mathcal{C}_1)$, $G(v)$ will be the unique morphism $u : G(X_2) \rightarrow G(Y_2)$ such that $F(u) = v$; G is then a *functor* from $F(\mathcal{C}_1)$ to \mathcal{C}_1 ; FG is the identity functor on $F(\mathcal{C}_1)$, and the above shows that there exists an isomorphism of functors $\phi : 1_{\mathcal{C}_1} \rightarrow GF$ such that F, G, ϕ , and the identity $1_{F(\mathcal{C}_1)} \rightarrow FG$ defines an *equivalence* between the category \mathcal{C}_1 and the full subcategory $F(\mathcal{C}_1)$ of \mathcal{C}_2 (T, 1.2).

(8.1.6). We apply Proposition (8.1.4) to the case where F is $h_{X'}$, X' being any object of \mathcal{C} ; the map $\beta : \mathrm{Hom}(X, X') \rightarrow \mathrm{Hom}(h_X, h_{X'})$ is none other than the map $w \mapsto h_w$ defined in (8.1.2); this map being *bijective*, we see with the terminology of (8.1.5) that:

Proposition (8.1.7). — *The canonical functor $h : \mathcal{C} \rightarrow \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ is fully faithful.*

(8.1.8). Let F be a contravariant functor from \mathcal{C} to Set ; we say that F is *representable* if there exists an object $X \in \mathcal{C}$ such that F is *isomorphic* to h_X ; it follows from Proposition (8.1.7) that the data of an $X \in \mathcal{C}$ and an isomorphism of functors $g : h_X \rightarrow F$ determines X up to unique isomorphism. Proposition (8.1.7) then implies that h defines an *equivalence* between \mathcal{C} and the full subcategory of $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ consisting of the *contravariant representable functors*. It follows from Proposition (8.1.4) that the data of a natural transformation $g : h_X \rightarrow F$ is equivalent to that of an element $\zeta \in F(X)$; to say that g is an *isomorphism* is equivalent to the following condition on ζ : *for every object Y of \mathcal{C} the map $v \mapsto (F(v))(\zeta)$ from $\mathrm{Hom}(Y, X)$ to $F(Y)$ is bijective*. When ζ satisfies this condition, we say that the pair (X, ζ) *represents* the representable functor F . By abuse of language, we also say that the object $X \in \mathcal{C}$ *represents* F if there exists a $\zeta \in F(X)$ such that (X, ζ) represents F , in other words if h_X is isomorphic to F .

Let F, F' be two contravariant representable functors from \mathcal{C} to Set , $h_X \rightarrow F$ and $h_{X'} \rightarrow F'$ two isomorphisms of functors. Then it follows from (8.1.6) that there is a canonical bijective correspondence between $\mathrm{Hom}(X, X')$ and the set $\mathrm{Hom}(F, F')$ of natural transformations $F \rightarrow F'$.

(8.1.9). *Example I. Projective limits.* The notion of a contravariant representable functor covers in particular the “dual” notion of the usual notion of a “solution to a universal problem”. More generally, we will see that the notion of the *projective limit* is a special case of the notion of a representable functor. Recall that in a category \mathcal{C} , we define a *projective system* by the data of a preordered set I , a family $(A_\alpha)_{\alpha \in I}$ of objects of \mathcal{C} , and for every pair of indices (α, β) such that $\alpha \leq \beta$, a morphism $u_{\alpha\beta} : A_\beta \rightarrow A_\alpha$. A *projective limit* of this system in \mathcal{C} consists of an object B of \mathcal{C} (denoted $\varprojlim A_\alpha$), and for each $\alpha \in I$, a morphism $u_\alpha : B \rightarrow A_\alpha$ such that: 1st. $u_\alpha = u_{\alpha\beta}u_\beta$ for $\alpha \leq \beta$; 2nd. for every object X of \mathcal{C} and every family $(v_\alpha)_{\alpha \in I}$ of morphisms $v_\alpha : X \rightarrow A_\alpha$ such that $v_\alpha = u_{\alpha\beta}v_\beta$ for $\alpha \leq \beta$, there exists a unique morphism $v : X \rightarrow B$ (denoted $\varprojlim v_\alpha$) such that $v_\alpha = u_\alpha v$ for all $\alpha \in I$ (T, 1.8). This can be interpreted in the following way: the $u_{\alpha\beta}$ canonically define maps

$$\bar{u}_{\alpha\beta} : \mathrm{Hom}(X, A_\beta) \longrightarrow \mathrm{Hom}(X, A_\alpha)$$

which define a *projective system* of sets $(\mathrm{Hom}(X, A_\alpha), \bar{u}_{\alpha\beta})$, and (v_α) is by definition an element of the set $\varprojlim \mathrm{Hom}(X, A_\alpha)$; it is clear that $X \mapsto \varprojlim \mathrm{Hom}(X, A_\alpha)$ is a *contravariant functor* from \mathcal{C} to Set , and the existence of the projective limit B is equivalent to saying that $(v_\alpha) \mapsto \varprojlim v_\alpha$ is an *isomorphism* of functors in X

$$(8.1.9.1) \quad \varprojlim \mathrm{Hom}(X, A_\alpha) \simeq \mathrm{Hom}(X, B),$$

in other words, that the functor $X \mapsto \varprojlim \mathrm{Hom}(X, A_\alpha)$ is *representable*.

(8.1.10). *Example II. Final objects.* Let \mathcal{C} be a category, $\{a\}$ a singleton set. Consider the contravariant functor $F : \mathcal{C} \rightarrow \text{Set}$ which sends every object X of \mathcal{C} to the set $\{a\}$, and every morphism $X \rightarrow X'$ in \mathcal{C} to the unique map $\{a\} \rightarrow \{a\}$. To say that this functor is *representable* means that there exists an object $e \in \mathcal{C}$ such that for every $Y \in \mathcal{C}$, $\text{Hom}(Y, e) = h_e(Y)$ is a *singleton set*; we say that e is a *final object* of \mathcal{C} , and it is clear that two final objects of \mathcal{C} are isomorphic (which allows us to define, in general with the axiom of choice, *one* final object of \mathcal{C} which we then denote $e_{\mathcal{C}}$). For example, in the category Set , the final objects are the singleton sets; in the category of *augmented algebras* over a field K (where the morphisms are the algebra homomorphisms compatible with the augmentation), K is a final object; in the category of *S-preschemes* (I, 2.5.1), S is a final object.

(8.1.11). For two objects X and Y of a category \mathcal{C} , set $h'_X(Y) = \text{Hom}(X, Y)$, and for every morphism $u : Y \rightarrow Y'$, let $h_X(u)$ be the map $v \mapsto vu$ from $\text{Hom}(X, Y)$ to $\text{Hom}(X, Y')$; h'_X is then a *covariant functor* $\mathcal{C} \rightarrow \text{Set}$, so we deduce as in (8.1.2) the definition of a canonical covariant functor $h' : \mathcal{C}^{\text{op}} \rightarrow \text{Hom}(\mathcal{C}, \text{Set})$; a *covariant functor* F from \mathcal{C} to Set , in other words an object of $\text{Hom}(\mathcal{C}, \text{Set})$, is then *representable* if there exists an object $X \in \mathcal{C}$ (necessarily unique up to unique isomorphism) such that F is *isomorphic to* h'_X ; we leave it to the reader to develop the “dual” notions of the above, which this time cover the notion of an *inductive limit*, and in particular the usual notion of a “solution to a universal problem”.

8.2. Algebraic structures in categories

(8.2.1). Given two contravariant functors F and F' from \mathcal{C} to Set , recall that for every object $Y \in \mathcal{C}$, we set $(F \times F')(Y) = F(Y) \times F'(Y)$, and for every morphism $u : Y \rightarrow Y'$ in \mathcal{C} , we set $(F \times F')(u) = F(u) \times F'(u)$, which is the map $(t, t') \mapsto (F(u)(t), F'(u)(t'))$ from $F(Y') \times F'(Y')$ to $F(Y) \times F'(Y)$; $F \times F' : \mathcal{C} \rightarrow \text{Set}$ is thus a *contravariant functor* (which is none other than the *product* of the objects F and F' in the category $\text{Hom}(\mathcal{C}^{\text{op}}, \text{Set})$). Given an object $X \in \mathcal{C}$, we call an *internal composition law* on X a *natural transformation*

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$$(8.2.1.1) \quad \gamma_X : h_X \times h_X \longrightarrow h_X.$$

In other words (T, 1.2), for every object $Y \in \mathcal{C}$, $\gamma_X(Y)$ is a map $h_X(Y) \times h_X(Y) \rightarrow h_X(Y)$ (thus by definition an *internal composition law* on the set $h_X(Y)$) with the condition that for every morphism $u : Y \rightarrow Y'$ in \mathcal{C} , the diagram

$$\begin{array}{ccc} h_X(Y') \times h_X(Y') & \xrightarrow{h_X(u) \times h_X(u)} & h_X(Y) \times h_X(Y) \\ \downarrow \gamma_X(Y') & & \downarrow \gamma_X(Y) \\ h_X(Y') & \xrightarrow{h_X(u)} & h_X(Y) \end{array}$$

is commutative; this implies that for the composition laws $\gamma_X(Y)$ and $\gamma_X(Y')$, $h_X(u)$ is a *homomorphism* from $h_X(Y')$ to $h_X(Y)$.

In a similar way, given two objects Z and X of \mathcal{C} , we call an *external composition law* on X , with Z as its domain of operators a natural transformation

$$(8.2.1.2) \quad \omega_{X,Z} : h_Z \times h_X \longrightarrow h_X.$$

We see as above that for every $Y \in \mathcal{C}$, $\omega_{X,Z}(Y)$ is an external composition law on $h_X(Y)$, with $h_Z(Y)$ as its domain of operators and such that for every morphism $u : Y \rightarrow Y'$, $h_X(u)$ and $h_Z(u)$ form a *di-homomorphism* from $(h_Z(Y'), h_X(Y'))$ to $(h_Z(Y), h_X(Y))$.

(8.2.2). Let X' be a second object of \mathcal{C} , and suppose we are given an internal composition law $\gamma_{X'}$ on X' ; we say that a morphism $w : X \rightarrow X'$ in \mathcal{C} is a *homomorphism* for the composition laws if for every $Y \in \mathcal{C}$, $h_w(Y) : h_X(Y) \rightarrow h_{X'}(Y)$ is a *homomorphism* for the composition laws $\gamma_X(Y)$ and $\gamma_{X'}(Y)$. If X'' is a third object of \mathcal{C} equipped with an internal composition law $\gamma_{X''}$ and $w' : X' \rightarrow X''$ is a morphism in \mathcal{C} which is a homomorphism for $\gamma_{X'}$ and $\gamma_{X''}$, then it is clear that the morphism $w'w : X \rightarrow X''$ is a homomorphism for the composition laws γ_X and $\gamma_{X''}$. An isomorphism $w : X \simeq X'$ in \mathcal{C} is called

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an isomorphism for the composition laws γ_X and $\gamma_{X'}$ if w is a homomorphism for these composition laws, and if its inverse morphism w^{-1} is a homomorphism for the composition laws $\gamma_{X'}$ and γ_X .

We define in a similar way the *di-homomorphisms* for pairs of objects of \mathcal{C} equipped with external composition laws.

(8.2.3). When an internal composition law γ_X on an object $X \in \mathcal{C}$ is such that $\gamma_X(Y)$ is a *group law* on $h_X(Y)$ for every $Y \in \mathcal{C}$, we say that X , equipped with this law, is a *\mathcal{C} -group* or a *group object in \mathcal{C}* . We similarly define *\mathcal{C} -rings*, *\mathcal{C} -modules*, etc.

(8.2.4). Suppose that the *product* $X \times X$ of an object $X \in \mathcal{C}$ by itself exists in \mathcal{C} ; by definition, we then have $h_{X \times X} = h_X \times h_X$ up to canonical isomorphism, since it is a particular case of the projective limit (8.1.9); an internal composition law on X can thus be considered as a functorial morphism $\gamma_X : h_{X \times X} \rightarrow h_X$, and thus canonically determine (8.1.6) an element $c_X \in \text{Hom}(X \times X, X)$ such that $h_{c_X} = \gamma_X$; in this case, the data of an internal composition law on X is equivalent to the data of a morphism $X \times X \rightarrow X$; when \mathcal{C} is the category *Set*, we recover the classical notion of an internal composition law on a set. We have an analogous result for an external composition law when the product $Z \times X$ exists in \mathcal{C} .

(8.2.5). With the above notation, suppose that in addition $X \times X \times X$ exists in \mathcal{C} ; the characterization of the product as an object representing a functor (8.1.9) implies the existence of canonical isomorphisms

$$(X \times X) \times X \simeq X \times X \times X \simeq X \times (X \times X);$$

if we canonically identify $X \times X \times X$ with $(X \times X) \times X$, then the map $\gamma_X(Y) \times 1_{h_X(Y)}$ identifies with $h_{c_X \times 1_X}(Y)$ for all $Y \in \mathcal{C}$. As a result, it is equivalent to say that for every $Y \in \mathcal{C}$, the internal law $\gamma_X(Y)$ is associative, or that the diagram of maps

$$\begin{array}{ccc} h_X(Y) \times h_X(Y) \times h_X(Y) & \xrightarrow{\gamma_X(Y) \times 1} & h_X(Y) \times h_X(Y) \\ \downarrow 1 \times \gamma_X(Y) & & \downarrow \gamma_X(Y) \\ h_X(Y) \times h_X(Y) & \xrightarrow{\gamma_X(Y)} & h_X(Y) \end{array}$$

is commutative, or that the diagram of morphisms

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$$\begin{array}{ccc} X \times X \times X & \xrightarrow{c_X \times 1_X} & X \times X \\ \downarrow 1_X \times c_X & & \downarrow c_X \\ X \times X & \xrightarrow{c_X} & X \end{array}$$

is commutative.

(8.2.6). Under the hypotheses of (8.2.5), if we want to express, for every $Y \in \mathcal{C}$, the internal law $\gamma_X(Y)$ as a *group law*, then it is first necessary that it is associative, and second that there exists a map $\alpha_X(Y) : h_X(Y) \rightarrow h_X(Y)$ having the properties of the *inverse* operation of a group; as for every morphism $u : Y \rightarrow Y'$ in \mathcal{C} , we have seen that $h_X(u)$ must be a group homomorphism $h_X(Y') \rightarrow h_X(Y)$, we first see that $\alpha_X : h_X \rightarrow h_X$ must be a *natural transformation*. On the other hand, one can express the characteristic properties of the inverse $s \mapsto s^{-1}$ of a group G without involving the identity element: it suffices to check that the two composite maps

$$\begin{aligned} (s, t) &\mapsto (s, s^{-1}, t) \mapsto (s, s^{-1}t) \mapsto s(s^{-1}t), \\ (s, t) &\mapsto (s, s^{-1}, t) \mapsto (s, ts^{-1}) \mapsto (ts^{-1})s \end{aligned}$$

are equal to the second projection $(s, t) \mapsto t$ from $G \times G$ to G . By (8.1.3), we have $\alpha_X = h_{a_X}$, where $a_X \in \text{Hom}(X, X)$; the first condition above then expresses that the composite morphism

$$X \times X \xrightarrow{(1_X, a_X) \times 1_X} X \times X \times X \xrightarrow{1_X \times c_X} X \times X \xrightarrow{c_X} X$$

is the second projection $X \times X \rightarrow X$ in \mathcal{C} , and the second condition is similar.

(8.2.7). Now suppose that there exists a *final object* e (8.1.10) in \mathcal{C} . Let us always assume that $\gamma_X(Y)$ is a group law on $h_X(Y)$ for every $Y \in \mathcal{C}$, and denote by $\eta_X(Y)$ the identity element of $\gamma_X(Y)$. As, for every morphism $u : Y \rightarrow Y'$ in \mathcal{C} , $h_X(u)$ is a group homomorphism, we have $\eta_X(Y) = (h_X(u))(\eta_X(Y'))$; taking in particular $Y' = e$, in which case u is the unique element ε of $\text{Hom}(Y, e)$, we see that the element $\eta_X(e)$ completely determines $\eta_X(Y)$ for every $Y \in \mathcal{C}$. Set $e_X = \eta_X(X)$, the identity element of the group $h_X(X) = \text{Hom}(X, X)$; the commutativity of the diagram

$$\begin{array}{ccc} h_X(e) & \xrightarrow{h_X(\varepsilon)} & h_X(Y) \\ h_{e_X}(e) \downarrow & & \downarrow h_{e_X}(Y) \\ h_X(e) & \xrightarrow{h_X(\varepsilon)} & h_X(Y) \end{array}$$

(cf. (8.1.2)) shows that, on the set $h_X(Y)$, the map $h_{e_X}(Y)$ is none other than $s \mapsto \eta_X(Y)$ sending every element to the identity element. We then verify that the fact that $\eta_X(Y)$ is the identity element of $\gamma_X(Y)$ for every $Y \in \mathcal{C}$ is equivalent to saying that the composite morphism

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$$X \xrightarrow{(1_X, 1_X)} X \times X \xrightarrow{1_X \times e_X} X \times X \xrightarrow{c_X} X,$$

and the analog in which we swap 1_X and e_X , are both *equal* to 1_X .

(8.2.8). One could of course easily extend the examples of algebraic structures in categories. The example of groups was treated with enough detail, but latter on we will usually leave it to the reader to develop analogous notions for the examples of algebraic structures we will encounter.

§9. CONSTRUCTIBLE SETS

9.1. Constructible sets

Definition (9.1.1). — We say that a continuous map $f : X \rightarrow Y$ is *quasi-compact* if for every quasi-compact open subset U of Y , $f^{-1}(U)$ is quasi-compact. We say that a subset Z of a topological space X is *retrocompact* in X if the canonical injection $Z \rightarrow X$ is quasi-compact, in other words, if for every quasi-compact open subset U of X , $U \cap Z$ is quasi-compact.

A *closed* subset of X is retrocompact in X , but a quasi-compact subset of X is not necessarily retrocompact in X . If X is quasi-compact, every retrocompact open subset of X is quasi-compact. It is clear that every *finite* union of retrocompact sets in X is retrocompact in X , as every finite union of quasi-compact sets is quasi-compact. Every finite intersection of retrocompact open sets in X is a retrocompact open set in X . In a *locally Noetherian* space X , every quasi-compact set is a Noetherian subspace, and as a result *every subset* of X is retrocompact in X .

Definition (9.1.2). — Given a topological space X , we say that a subset of X is *constructible* if it belongs to the smallest set of subsets \mathfrak{F} of X containing all the retrocompact open subsets of \mathfrak{F} and is stable under finite intersections and complements (which implies that \mathfrak{F} is also stable under finite unions).

Proposition (9.1.3). — For a subset of X to be constructible, it is necessary and sufficient for it to be a finite union of sets of the form $U \cap \mathbb{C}V$, where U and V are retrocompact open sets in X .

Proof. It is clear that the condition is sufficient. To see that it is necessary, consider the set \mathfrak{G} of finite unions of sets of the form $U \cap \mathbb{C}V$, where U and V are retrocompact open sets in X ; it suffices to see that every complement of a set in \mathfrak{G} is in \mathfrak{G} . Let $Z = \bigcup_{i \in I} (U_i \cap \mathbb{C}V_i)$, where I is finite, U_i and V_i retrocompact open sets in X ; we have $\mathbb{C}Z = \bigcap_{i \in I} (V_i \cup \mathbb{C}U_i)$, so Z is a finite union of sets which are intersections of a certain number of the V_i and of a certain number of the $\mathbb{C}U_i$, thus of the form $V \cap \mathbb{C}U$, where U is the union of a certain number of the U_i and V is the intersection of a certain number of the V_i ; but we have noted above that finite unions and intersections of retrocompact open sets in X are retrocompact open sets in X , hence the conclusion. \square

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Corollary (9.1.4). — Every constructible subset of X is retrocompact in X .

Proof. It suffices to show that if U and V are retrocompact open sets in X , then $U \cap \mathbb{C}V$ is retrocompact in X ; if W is a quasi-compact open set in X , then $W \cap U \cap \mathbb{C}V$ is closed in the quasi-compact space $W \cap U$, hence it is quasi-compact. \square

In particular:

Corollary (9.1.5). — *For an open subset U of X to be constructible, it is necessary and sufficient for it to be retrocompact in X . For a closed subset F of X to be constructible, it is necessary and sufficient for the open set $\mathbb{C}F$ to be retrocompact.*

(9.1.6). An important case is when every quasi-compact open subset of X is retrocompact, in other words, when the intersection of two quasi-compact open subsets of X is quasi-compact (cf. (I, 5.5.6)). When X is also quasi-compact, this implies that the retrocompact open subsets of X are identical to the quasi-compact open subsets of X , and the constructible subsets of X are finite unions of sets of the form $U \cap \mathbb{C}V$, where U and V are quasi-compact open sets.

Corollary (9.1.7). — *For a subset of a Noetherian space to be constructible, it is necessary and sufficient for it to be a finite union of locally closed subsets of X .*

Proposition (9.1.8). — *Let X be a topological space, U an open subset of X .*

- (i) *If T is a constructible subset of X , then $T \cap U$ is a constructible subset of U .*
- (ii) *In addition, suppose that U is retrocompact in X . For a subset Z of U to be constructible in X , it is necessary and sufficient for it to be constructible in U .*

Proof.

- (i) Using Proposition (9.1.3), we reduce to showing that if T is a retrocompact open set in X , then $T \cap U$ is a retrocompact open set in U , in other words, for every quasi-compact open $W \subset U$, $T \cap U \cap W = T \cap W$ is quasi-compact, which immediately follows from the hypothesis.
- (ii) The condition is necessary by (i), so it remains to show that it is sufficient. By Proposition (9.1.3), it suffices to consider the case where Z is a retrocompact open set in U , because it will then follow that $U - Z$ is constructible in X , and if Z and Z' are two retrocompact opens in U , then $Z \cap (U - Z')$ will be constructible in X . If W is a quasi-compact open set in X and Z a retrocompact open set in U , then we have $Z \cap W = Z \cap (W \cap U)$, and by hypothesis $W \cap U$ is a quasi-compact open set in U ; so $W \cap Z$ is quasi-compact, and as a result Z is a retrocompact open set in X , and *a fortiori* constructible in X . \square

Corollary (9.1.9). — *Let X be a topological space, $(U_i)_{i \in I}$ a finite cover of X by retrocompact open sets in X . For a subset Z of X to be constructible in X , it is necessary and sufficient for $Z \cap U_i$ to be constructible in U_i for all $i \in I$.*

(9.1.10). In particular, suppose that X is quasi-compact and every point of X admits a fundamental system of retrocompact open neighbourhoods in X (and *a fortiori* quasi-compact); then the condition for a subset Z of X to be constructible in X is of a *local* nature, in other words, it is necessary and sufficient that for every $x \in X$, there exists an open neighbourhood V of x such that $V \cap Z$ is constructible in V . Indeed, if this condition is satisfied, then there exists for every $x \in X$ an open neighbourhood V of x which is *retrocompact in X* and such that $V \cap Z$ is constructible in V , by the hypotheses on X and by Proposition (9.1.8, i); it then suffices to cover X by a finite number of these neighbourhoods and to apply Corollary (9.1.9).

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Definition (9.1.11). — Let X be a topological space. We say that a subset T of X is *locally constructible* in X if for every $x \in X$ there exists an open neighbourhood V of x such that $T \cap V$ is constructible in V .

It follows from Proposition (9.1.8, i) that if V is such that $V \cap T$ is constructible in V , then for every open $W \subset V$, $W \cap T$ is constructible in W . If T is locally constructible in X , then for every open set U in X , $T \cap U$ is locally constructible in U , as a result of the above remark. The same remark shows that the set of locally constructible subsets of X is stable under finite unions and finite intersections; on the other hand, it is clear that it is also stable under taking complements.

Proposition (9.1.12). — *Let X be a topological space. Every constructible set in X is locally constructible in X . The converse is true if X is quasi-compact and if its topology admits a basis formed by the retrocompact sets in X .*

Proof. The first assertion follows from Definition (9.1.11) and the second from (9.1.10). \square

Corollary (9.1.13). — *Let X be a topological space whose topology admits a basis formed by the retrocompact sets in X . Then every locally constructible subset T of X is retrocompact in X .*

Proof. Let U be a quasi-compact open set in X ; $T \cap U$ is locally constructible in U , hence constructible in U by Proposition (9.1.12), and as a result quasi-compact by Corollary (9.1.4). \square

9.2. Constructible subsets of Noetherian spaces

(9.2.1). We have seen (9.1.7) that in a Noetherian space X , the constructible subsets of X are the *finite unions of locally closed subsets of X* .

The inverse image of a constructible set in X by a continuous map from a Noetherian space X' to X is constructible in X' . If Y is a constructible subset of a Noetherian space X , then the subsets of Y are constructible as subspaces of Y and are identical to those which are constructible as subspaces of X .

Proposition (9.2.2). — *Let X be an irreducible Noetherian space, E a constructible subset of X . For E to be everywhere dense in X , it is necessary and sufficient for E to contain a nonempty open subset of X .*

Proof. The condition is evidently sufficient, as every nonempty open set is dense in X . Conversely, let $E = \bigcup_{i=1}^n (U_i \cap F_i)$ be a constructible subset of X , the U_i being nonempty open sets and the F_i closed in X ; we then have $\bar{E} \subset \bigcup_i F_i$. As a result, if $\bar{E} = X$, then X is equal to one of the F_i , hence $E \supset U_i$, which finishes the proof. \square

When X admits a generic point x (0, 2.1.2), the condition of Proposition (9.2.2) is equivalent to the relation $x \in E$.

Proposition (9.2.3). — *Let X be a Noetherian space. For a subset E of X to be constructible, it is necessary and sufficient that, for every irreducible closed subset Y of X , $E \cap Y$ is rare in Y or contains a nonempty open subset of Y .*

Proof. The necessity of the condition follows from the fact that $E \cap Y$ must be a constructible subset of Y and from Proposition (9.2.2), since a nondense subset of Y is necessarily rare in the irreducible space Y (0, 2.1.1). To prove that the condition is sufficient, apply the principle of Noetherian induction (0, 2.2.2) to the set \mathfrak{F} of closed subsets Y of X such that $Y \cap E$ is constructible (with respect to Y or with respect to X , which are equivalent): we can thus assume that for every closed subset $Y \neq X$ of X , $E \cap Y$ is constructible. First suppose that X is not irreducible, and let X_i ($1 \leq i \leq m$) are its irreducible components, necessarily of finite number (0, 2.2.5); by hypothesis the $E \cap X_i$ are constructible, hence their union E is as well. Suppose now that X is irreducible; then by hypothesis, if E is rare, then $\bar{E} \neq X$ and $E = E \cap \bar{E}$ is constructible; if E contains a nonempty open subset U of X , then it is the union of U and $E \cap (X - U)$; but $X - U$ is a closed set distinct from X , so $E \cap (X - U)$ is constructible; as a result, E is itself constructible, which finishes the proof. \square

Corollary (9.2.4). — *Let X be a Noetherian space, (E_α) an increasing filtered family of constructible subsets of X , such that*

(1st) X is the union of the family (E_α) .

(2nd) Every irreducible closed subset of X is contained in the closure of one of the E_α .

Then there exists an index α such that $X = E_\alpha$.

When every irreducible closed subset of X admits a generic point, the hypothesis (1st) can be omitted.

Proof. We apply the principle of Noetherian induction (0, 2.2.2) to the set \mathfrak{M} of closed subsets of X contained in at least one of the E_α ; we can thus suppose that every closed subset $Y \neq X$ of X is contained in one of the E_α . The proposition is evident if X is not irreducible, because each of the irreducible components X_i of X ($1 \leq i \leq m$) is contained in an E_{α_i} , and there exists an E_α containing all of the E_{α_i} . Now suppose that X is irreducible. By hypothesis, there exists a β such that $X = \bar{E}_\beta$,

so (9.2.2) E_β contains a nonempty open subset U of X . But then the closed set $X - U$ is contained in an E_γ , and it suffices to take an E_α containing E_β and E_γ . When every irreducible closed subset Y of X admits a generic point y , there exists α such that $y \in E_\alpha$, so $Y = \overline{\{y\}} \subset \overline{E_\alpha}$, and condition (2nd) is therefore a consequence of (1st). \square

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Proposition (9.2.5). — *Let X be a Noetherian space, x a point of X , and E a constructible subset of X . For E to be a neighbourhood of x , it is necessary and sufficient that for every irreducible closed subset Y of X containing x , $E \cap Y$ is dense in Y (if there exists a generic point y of Y , this also implies (9.2.2) that $y \in E$).*

Proof. The condition is evidently necessary; we will prove that it is sufficient. Applying the principle of Noetherian induction to the set \mathfrak{M} of closed subsets Y of X containing x and such that $E \cap Y$ is a neighbourhood of x in Y , we can assume that every closed subset $Y \neq X$ of X containing x belongs to \mathfrak{M} . If X is not irreducible, then each of the irreducible components X_i of X containing x are distinct from X , hence $E \cap X_i$ is a neighbourhood of x with respect to X_i ; as a result, E is a neighbourhood of x in the union of the irreducible components of X containing x , and as this union is a neighbourhood of x in X , so is E . If X is irreducible, then E is dense in X by hypothesis, so it contains a nonempty open subset U of X (9.2.2); the proposition is then evident if $x \in U$; otherwise, x is by hypothesis inside $E \cap (X - U)$ with respect to $X - U$, so the closure of $X - E$ in X does not contain x , and the complement of this closure is a neighbourhood of x contained in E , which finishes the proof. \square

Corollary (9.2.6). — *Let X be a Noetherian space, E a subset of X . For E to be an open set in X , it is necessary and sufficient that for every irreducible closed subset Y of X intersecting E , $E \cap Y$ contains a nonempty open subset of Y .*

Proof. The condition is evidently necessary; conversely, if it is satisfied, then it implies that E is constructible by Proposition (9.2.3). In addition, Proposition (9.2.5) shows that E is then a neighbourhood of each of its points, hence the conclusion. \square

9.3. Constructible functions

Definition (9.3.1). — *Let h be a map from a topological space X to a set T . We say that h is constructible if $h^{-1}(t)$ is constructible for every $t \in T$, and empty except for finitely many values of t ; then for every subset S of T , $h^{-1}(S)$ is constructible. We say that h is locally constructible if every $x \in X$ has an open neighbourhood V such that $h|_V$ is constructible.*

Every constructible function is locally constructible; the converse is true when X is quasi-compact and admits a basis formed by the retrocompact open sets in X (in particular, when X is Noetherian).

Proposition (9.3.2). — *Let h be a map from a Noetherian space X to a set T . For h to be constructible, it is necessary and sufficient that for every irreducible closed subset Y of X , there exists a nonempty subset U of Y , open with respect to Y , in which h is constant.*

Proof. The condition is necessary: indeed, by hypothesis, h does not take finitely many values t_i on Y , and each of the sets $h^{-1}(t_i) \cap Y$ is constructible in Y (9.2.1); as they can not all be rare subsets of the space Y , at least one of them contains a nonempty open set (9.2.3).

To see that the condition is sufficient, we apply the principle of Noetherian induction on the set \mathfrak{M} of closed subsets Y of X such that the restriction $h|_Y$ is constructible; we can thus assume that for every closed subset $Y \neq X$ of X , $h|_Y$ is constructible. If X is not irreducible, then the restriction of h to each of the (finitely many) irreducible components X_i of X is constructible, and it then follows immediately from Definition (9.3.1) that h is constructible. If X is irreducible, then there exists by hypothesis a nonempty open subset U of X on which h is constant; on the other hand, the restriction of h to $X - U$ is constructible by hypothesis, and it follows immediately that h is constructible. \square

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Corollary (9.3.3). — *Let X be a Noetherian space in which every irreducible closed subset admits a generic point. If h is a map from X to a set T such that, for every $t \in T$, $h^{-1}(t)$ is constructible, then h is constructible.*

Proof. If Y is an irreducible closed subset of X and y its generic point, then $Y \cap h^{-1}(h(y))$ is constructible and contains y , hence (9.2.2) this set contains a nonempty open subset of Y , and it suffices to apply Proposition (9.3.2). \square

Proposition (9.3.4). — *Let X be a Noetherian space in which every irreducible closed subset admits a generic point, h a constructible map from X to an ordered set. For h to be upper semi-continuous on X , it is necessary and sufficient that for every $x \in X$ and every specialization (0, 2.1.2) x' of x , we have $h(x') \leq h(x)$.*

Proof. The function h does not take a finite number of values; therefore, to say that it is upper semi-continuous means that for every $x \in X$, the set E of the $y \in X$ such that $h(y) \leq h(x)$ is a neighbourhood of x . By hypothesis, E is a constructible subset of X ; on the other hand, to say that an irreducible closed subset Y of X contains x means that its generic point y is a specialization of x ; the conclusion then follows from Proposition (9.2.5). \square

§10. SUPPLEMENT ON FLAT MODULES

For any proofs missing in (10.1) and (10.2), we refer the reader to Bourbaki, *Alg. comm.*, chap. II and III.

10.1. Relations between flat modules and free modules

(10.1.1). Let A be a ring, \mathfrak{J} an ideal of A , and M an A -module; for every integer $p \geq 0$, we have a canonical homomorphism of (A/\mathfrak{J}) -modules

$$(10.1.1.1) \quad \phi_p : (M/\mathfrak{J}M) \otimes_{A/\mathfrak{J}} (\mathfrak{J}^p/\mathfrak{J}^{p+1}) \longrightarrow \mathfrak{J}^p M/\mathfrak{J}^{p+1}M,$$

which is evidently *surjective*. We denote by $\text{gr}(A) = \bigoplus_{p \geq 0} \mathfrak{J}^p/\mathfrak{J}^{p+1}$ the graded ring associated to A filtered by the \mathfrak{J}^p , and by $\text{gr}(M) = \bigoplus_{p \geq 0} \mathfrak{J}^p M/\mathfrak{J}^{p+1}M$ the graded $\text{gr}(A)$ -module associated to M filtered by the $\mathfrak{J}^p M$; we then have $\text{gr}_p(A) = \mathfrak{J}^p/\mathfrak{J}^{p+1}$, and $\text{gr}_p(M) = \mathfrak{J}^p M/\mathfrak{J}^{p+1}M$; the ϕ define a *surjective* homomorphism of graded $\text{gr}(A)$ -modules

$$(10.1.1.2) \quad \phi : \text{gr}_0(M) \otimes_{\text{gr}_0(A)} \text{gr}(A) \longrightarrow \text{gr}(M).$$

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(10.1.2). Suppose that *one* of the following hypotheses is satisfied:

- (i) \mathfrak{J} is nilpotent;
- (ii) A is Noetherian, \mathfrak{J} is contained in the radical of A , and M is of finite type.

Then the following properties are equivalent.

- (a) M is a free A -module.
- (b) $M/\mathfrak{J}M = N \otimes_A (A/\mathfrak{J})$ is a free (A/\mathfrak{J}) -module, and $\text{Tor}_1^A(M, A/\mathfrak{J}) = 0$.
- (c) $M/\mathfrak{J}M$ is a free (A/\mathfrak{J}) -module, and the canonical homomorphism (10.1.1.2) is injective (and thus bijective).

(10.1.3). Suppose that A/\mathfrak{J} is a *field* (in other words, that \mathfrak{J} is maximal), and that one of the hypotheses, (i) and (ii), of (10.1.2) is satisfied. Then the following properties are equivalent.

- (a) M is a free A -module.
- (b) M is a projective A -module.
- (c) M is a flat A -module.
- (d) $\text{Tor}_1^A(M, A/\mathfrak{J}) = 0$.
- (e) The canonical homomorphism (10.1.1.2) is bijective.

This result can be applied, in particular, to the following two cases:

- (i) M is an *arbitrary* module, over a local ring A whose maximal ideal \mathfrak{J} is *nilpotent* (for example, a local Artinian ring);
- (ii) M is a module of *finite type* over a *local Noetherian* ring.

10.2. Local flatness criteria

(10.2.1). With the hypotheses and notation of (10.1.1), consider the following conditions.

- (a) M is a flat A -module.
- (b) $M/\mathfrak{J}M$ is a flat (A/\mathfrak{J}) -module, and $\mathrm{Tor}_1^A(M, A/\mathfrak{J}) = 0$.
- (c) $M/\mathfrak{J}M$ is a flat (A/\mathfrak{J}) -module, and the canonical homomorphism (10.1.1.2) is bijective.
- (d) For all $n \geq 1$, $M/\mathfrak{J}^n M$ is a flat (A/\mathfrak{J}^n) -module.

Then we have the implications

$$(a) \implies (b) \implies (c) \implies (d),$$

and, if \mathfrak{J} is *nilpotent*, then the four conditions are *equivalent*. This is also the case if A is Noetherian and M is *ideally separated*, that is to say, for every ideal \mathfrak{a} of A , the A -module $\mathfrak{a} \otimes_A M$ is *separated* for the \mathfrak{J} -preadic topology.

(10.2.2). Let A be a Noetherian ring, B a commutative Noetherian A -algebra, \mathfrak{J} an ideal of A such that $\mathfrak{J}B$ is contained in the radical of B , and M a B -module of finite type. Then, when M is considered as an A -module, the four conditions of (10.2.1) are *equivalent*. This result applies first and foremost in the case where A and B are *local* Noetherian rings, with the homomorphism $A \rightarrow B$ being a *local* homomorphism. More specifically, if \mathfrak{J} is then the *maximal* ideal of A , we can, in conditions (b) and (c), remove the hypothesis that $M/\mathfrak{J}M$ is flat, since it is automatically satisfied, and condition (d) implies that the modules $M/\mathfrak{J}^n M$ are *free* over the A/\mathfrak{J}^n .

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(10.2.3). With the hypotheses on A , B , \mathfrak{J} , and M from the start of (10.2.2), let \widehat{A} be the separated completion of A for the \mathfrak{J} -preadic topology, and \widehat{M} the separated completion of M for the $\mathfrak{J}B$ -preadic topology. Then, for M to be a flat A -module, it is necessary and sufficient for \widehat{M} to be a flat \widehat{A} -module.

(10.2.4). Let $\rho : A \rightarrow B$ be a local homomorphism of local Noetherian rings, k the residue field of A , and M and N both B -modules of finite type, with N assumed to be *A-flat*. Let $u : M \rightarrow N$ be a B -homomorphism. Then the following conditions are equivalent.

- (a) u is injective, and $\mathrm{Coker}(u)$ is a flat A -module.
- (b) $u \otimes 1 : M \otimes_A k \rightarrow N \otimes_A k$ is injective.

(10.2.5). Let $\rho : A \rightarrow B$ and $\sigma : B \rightarrow C$ be local homomorphisms of local Noetherian rings, k the residue field of A , and M a C -module of finite type. Suppose that B is a *flat* A -module. Then the following conditions are equivalent.

- (a) M is a flat B -module.
- (b) M is a flat A -module, and $M \otimes_A k$ is a flat $(B \otimes_A k)$ -module.

Proposition (10.2.6). — *Let A and B be local Noetherian rings, $\rho : A \rightarrow B$ a local homomorphism, \mathfrak{J} an ideal of B contained in the maximal ideal, and M a B -module of finite type. Suppose that, for all $n \geq 0$, $M_n = M/\mathfrak{J}^{n+1}M$ is a flat A -module. Then M is a flat A -module.*

Proof. We have to prove that, for every injective homomorphism $u : N' \rightarrow N$ of A -modules of finite type, $v = 1 \otimes u : M \otimes_A N' \rightarrow M \otimes_A N$ is injective. But $M \otimes_A N'$ and $M \otimes_A N$ are B -modules of finite type, and thus separated for the \mathfrak{J} -preadic topology (0I, 7.3.5); it thus suffices to prove that the homomorphism $\widehat{v} : \widehat{M \otimes_A N'} \rightarrow \widehat{M \otimes_A N}$ of the separated completions is injective. But $\widehat{v} = \varprojlim v_n$, where v_n is the homomorphism $1 \otimes u : M_n \otimes_A N' \rightarrow M_n \otimes_A N$; since, by hypothesis, M_n is A -flat, v_n is injective for all n , and thus so too is v , because the functor \varprojlim is left exact. \square

Corollary (10.2.7). — *Let A be a Noetherian ring, B a local Noetherian ring, $\rho : A \rightarrow B$ a homomorphism, f an element of the maximal ideal of B , and M a B -module of finite type. Suppose that the homothety $f_M : x \rightarrow fx$ on M is injective, and that M/fM is a flat A -module. Then M is a flat A -module.*

Proof. Let $M_i = f^i M$ for $i \geq 0$; since f_M is injective, M_i/M_{i+1} is isomorphic to M/fM , and thus A -flat for all $i \geq 0$; the exact sequence

$$0 \longrightarrow M_i/M_{i+1} \longrightarrow M/M_{i+1} \longrightarrow M/M_i \longrightarrow 0$$

gives us, by induction on i , that M/M_i is *A-flat* for all $i \geq 0$ (0I, 6.1.2); we can thus apply (10.2.6). We can also argue directly as follows: for every A -module N of finite type, $M \otimes_A N$ is a B -module of

finite type; since f belongs to the radical \mathfrak{n} of B , the (f) -adic topology on $M \otimes_A N$ is finer than the \mathfrak{u} -adic topology, and we know that the latter is *separated* (0_I, 0.7.3.5,) Now, since M/M_i is A -flat, we have that

$$f^i(M \otimes_A N) = \text{Im}(M_i \otimes_A N \longrightarrow M \otimes_A N) = \text{Ker}(M \otimes_A N \longrightarrow (M/M_i) \otimes_A N)$$

by (0_I, 6.1.2). So let N be an A -module of finite type, and N' a submodule of N , with canonical injection $j : N' \rightarrow N$; in the commutative diagram

$$\begin{array}{ccc} M \otimes_A N' & \longrightarrow & (M/M_i) \otimes_A N' \\ 1_{M \otimes j} \downarrow & & \downarrow 1_{M/M_i} \otimes j \\ M \otimes_A N & \longrightarrow & (M/M_i) \otimes_A N \end{array}$$

$1_{M/M_i} \otimes j$ is injective, because M/M_i is A -flat; we thus conclude that

$$\text{Ker}(M \otimes_A N' \longrightarrow M \otimes_A N) \subset \text{Ker}(M \otimes_A N' \longrightarrow (M/M_i) \otimes_A N')$$

for any i ; since the intersection (over i) of the latter kernel is 0, as we saw above, so too is the intersection (over i) of the former, and so M is A -flat. \square

Proposition (10.2.8). — *Let A be a reduced Noetherian ring, and M an A -module of finite type. Suppose that, for every A -algebra B (which is then a discrete valuation ring), $M \otimes_A B$ is a flat B -module (and thus free (10.1.3)). Then M is a flat A -module.*

Proof. We know that, for M to be flat, it is necessary and suffices for $M_{\mathfrak{m}}$ to be a flat $A_{\mathfrak{m}}$ -module for every maximal ideal \mathfrak{m} of A (0_I, 6.3.3); we can thus restrict to the case where A is *local* (0_I, 1.2.8). So let \mathfrak{m} be the maximal ideal of A , \mathfrak{p}_i ($1 \leq i \leq r$) the minimal prime ideals of A , and k the residue field A/\mathfrak{m} . We know (II, 7.1.7) that there exists, for each i , a discrete valuation ring B_i that has the same field of fractions K_i as the integral ring A/\mathfrak{p}_i , and that, further, dominates A/\mathfrak{p}_i . Let $M_i = M \otimes_A B_i$. By hypothesis, M_i is free over B_i , and so, denoting by k_i the residue field of B_i , we have

$$(10.2.8.1) \quad \text{rg}_{k_i}(M_i \otimes_{B_i} k_i) = \text{rg}_{K_i}(M_i \otimes_{B_i} K_i).$$

But it is clear that the composite homomorphism $A \rightarrow A/\mathfrak{p}_i \rightarrow B_i$ is local, and so k is an extension of k_i , and that we have $M_i \otimes_{B_i} k_i = M \otimes_A k_i = (M \otimes_A k) \otimes k_i$, and also that $M_i \otimes_{B_i} K_i = M \otimes_A K_i$. Equation (10.2.8.1) thus implies that

$$\text{rg}_k(M \otimes_A k) = \text{rg}_{K_i}(M \otimes_A K_i) \quad \text{for } 1 \leq i \leq r$$

and since A is reduced, we know that this condition implies that M is a *free* A -module (Bourbaki, *Alg. comm.*, chap. II, § 3, n° 2, prop. 7). \square

10.3. Existence of flat extensions of local rings

Proposition (10.3.1). — *Let A be a local Noetherian ring, with maximal ideal \mathfrak{J} , and residue field $k = A/\mathfrak{J}$. Let K be a field extension of k . Then there exists a local homomorphism from A to a local Noetherian ring B , such that $B/\mathfrak{J}B$ is k -isomorphic to K , and such that B is a flat A -module.*

The rest of this section is devoted to proving this proposition, step-by-step.

(10.3.1.1). First suppose that $K = k(T)$, where T is an indeterminate. In the ring of polynomials $A' = A[T]$, consider the prime ideal $\mathfrak{J}' = \mathfrak{J}A$, consisting of the polynomials that have coefficients in the ideal \mathfrak{J} ; it is clear that A'/\mathfrak{J}' is canonically isomorphic to $k[T]$. We will show that the ring of fractions $B = A'_{\mathfrak{J}'}$ is that for which we are searching (that is, a ring which satisfies the conditions of the conclusion of the proposition); it is clearly a local Noetherian ring, with maximal ideal $\mathfrak{L} = \mathfrak{J}B$. Further, $B/\mathfrak{L} = (A'/\mathfrak{J}')_{\mathfrak{J}'} = (k[T])_{\mathfrak{J}'}$ is exactly the field of fractions K of $k[T]$. Finally, B is a flat A' -module, and A' is a free A -module, so B is a flat A -module (0_I, 6.2.1).

(10.3.1.2). Now suppose that $K = k(t) = k[t]$, where t is algebraic over k ; let $f \in k[T]$ be the minimal polynomial of t ; there exists a monic polynomial $F \in A[T]$ whose canonical image in $k[T]$ is f . So let $A' = A[T]$, and let \mathfrak{J}' be the ideal $\mathfrak{J}A' + (F)$ in A' . We will see that the quotient ring $B = A'/\mathfrak{J}'$ is

that for which we are searching. First of all, it is clear that B is a *free* A -module, and thus flat. The ring A'/\mathfrak{J}' is isomorphic to

$$(A'/\mathfrak{J}A')/((\mathfrak{J}A' + (F))/\mathfrak{J}A') = k[T]/(f) = K;$$

the image \mathfrak{L} of \mathfrak{J}' in B is thus maximal, and we evidently have that $\mathfrak{L} = \mathfrak{J}B$. Finally, B is a semi-local ring, because it is an A -module of finite type (Bourbaki, *Alg. comm.*, chap. IV, § 2, n° 5, cor. 3 of prop. 9), and its maximal ideals are in bijective correspondence with those of $B/\mathfrak{J}B$ ([SZ60, vol. I, p. 259]); the previous arguments then prove that B is a local ring.

Lemma (10.3.1.3). — *Let $(A_\lambda, f_{\mu\lambda})$ be a filtered inductive system of local rings, such that the $f_{\mu\lambda}$ are local homomorphisms; let \mathfrak{m}_λ be the maximal ideal of A_λ , and let $K_\lambda = A_\lambda/\mathfrak{m}_\lambda$. Then $A' = \varinjlim A_\lambda$ is a local ring, with maximal ideal $\mathfrak{m} = \varinjlim \mathfrak{m}_\lambda$, and residue field $K = \varinjlim K_\lambda$. Further, if $\mathfrak{m}_\mu = \mathfrak{m}_\lambda A_\mu$ with $\lambda < \mu$, then we have $\mathfrak{m}' = \mathfrak{m}_\lambda A'$ for all λ . If, further, for $\lambda < \mu$, A_μ is a flat A_λ -module, and if all the A_λ are Noetherian, then A' is a flat Noetherian A_λ -modules for all λ .*

Proof. Since, by hypothesis, $(f_{\mu\lambda})(\mathfrak{m}_\lambda) \subset \mathfrak{m}_\mu$ for $\lambda < \mu$, the \mathfrak{m}_λ form an inductive system, and its limit \mathfrak{m}' is evidently an ideal of A' . Further, if $x' \notin \mathfrak{m}'$, there exists a λ such that $x' = f_\lambda(x_\lambda)$ for some $x_\lambda \in A_\lambda$ (where $f_\lambda : A_\lambda \rightarrow A'$ denotes the canonical homomorphism); because $x' \notin \mathfrak{m}'$, we necessarily have that $x_\lambda \notin \mathfrak{m}_\lambda$, and so x_λ admits an inverse y_λ in A_λ , and $y' = f_\lambda(y_\lambda)$ is the inverse of x' in A' , which proves that A' is a local ring with maximal ideal \mathfrak{m}' ; the claim about K follows immediately from the fact that \varinjlim is an exact functor. The hypothesis that $\mathfrak{m}_\mu = \mathfrak{m}_\lambda A_\mu$ implies that the canonical map $\mathfrak{m}_\lambda \otimes_{A_\lambda} A_\mu \rightarrow \mathfrak{m}_\mu$ is surjective; the equality $\mathfrak{m}' = \mathfrak{m}_\lambda A'$ then follows from, again, the fact that the functor \varinjlim is exact and commutes with the tensor product.

Now suppose that, for $\lambda < \mu$, we have $\mathfrak{m}_\mu = \mathfrak{m}_\lambda A_\mu$, and that A_μ is a flat A_λ -module. Then A' is a flat A_λ -module for all λ , by (0I, 6.2.3); since A' and A_λ are local rings, and since $\mathfrak{m}' = \mathfrak{m}_\lambda A'$, A' is even a *faithfully flat* A_λ -module (0I, 6.6.2). Finally, suppose further that the A_λ are *Noetherian*; the \mathfrak{m}_λ -adic topologies are then separated (0I, 7.3.5); we now show that, from this, it follows that the \mathfrak{m}' -adic topology on A' is *separated*. Indeed, if $x' \in A'$ belongs to all the \mathfrak{m}'^n ($n \geq 0$), then it is the image of some $x_\mu \in A_\mu$ for a specific index μ , and since the inverse image in A_μ of $\mathfrak{m}'^n = \mathfrak{m}_\mu^n A'$ is \mathfrak{m}_μ^n (0I, 6.6.1), x_μ belongs to all the \mathfrak{m}_μ^n , so $x_\mu = 0$, by hypothesis, and so $x' = 0$. Denote by \widehat{A}' the completion of A' for the \mathfrak{m}' -adic topology; the above shows that we have $A' \subset \widehat{A}'$. We will now show that \widehat{A}' is *Noetherian* and A_λ -*flat* for all λ ; from this, it will follow that \widehat{A}' is A' -flat (0I, 6.2.3), and since $\mathfrak{m}'\widehat{A}' \neq \widehat{A}'$, that \widehat{A}' is a *faithfully flat* A' -module (0I, 6.6.2), whence the final conclusion that A' is *Noetherian* (0I, 6.5.2), which will finish the proof of the lemma.

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We have $\widehat{A}' = \varprojlim_n A'/\mathfrak{m}'^n$; by the fact that A' is A_λ -flat, we have that

$$\mathfrak{m}^n/\mathfrak{m}^{n+1} = (\mathfrak{m}_\lambda^n/\mathfrak{m}_\lambda^{n+1}) \otimes_{A_\lambda} A' = (\mathfrak{m}_\lambda^n/\mathfrak{m}_\lambda^{n+1}) \otimes_{K_\lambda} (K_\lambda \otimes_{A_\lambda} A') = (\mathfrak{m}_\lambda^n/\mathfrak{m}_\lambda^{n+1}) \otimes_{K_\lambda} K;$$

since $\mathfrak{m}_\lambda^n/\mathfrak{m}_\lambda^{n+1}$ is a K_λ -vector space of finite dimension, $\mathfrak{m}_\lambda^n/\mathfrak{m}_\lambda^{n+1}$ is a K -vector space of finite dimension for all $n \geq 0$. It thus follows from (0I, 7.2.12) and (0I, 7.2.8) that \widehat{A}' is *Noetherian*. We further know that the maximal ideal of \widehat{A}' is $\mathfrak{m}'A'$, and that $\widehat{A}'/\mathfrak{m}'^n\widehat{A}'$ is isomorphic to A'/\mathfrak{m}'^n ; since $A'/\mathfrak{m}'^n = (A_\lambda/\mathfrak{m}_\lambda^n) \otimes_{A_\lambda} A'$, we see that A'/\mathfrak{m}'^n is a flat $(A_\lambda/\mathfrak{m}_\lambda^n)$ -module (0I, 6.2.1); criterion (10.2.2) is thus applicable to the Noetherian A_λ -algebra \widehat{A}' , and shows that \widehat{A}' is A_λ -flat. \square

(10.3.1.4). We now treat the general case. There exists an ordinal γ and, for every ordinal $\lambda \leq \gamma$, a subfield k_λ of K that contains k , such that (i) for all $\lambda < \gamma$, $k_{\lambda+1}$ is an extension of k_λ generated by a single element; (ii) for every limit ordinal μ , $k_\mu = \bigcup_{\lambda < \mu} k_\lambda$; and (iii) $K = k_\gamma$. In fact, it suffices to consider a bijection $\zeta \mapsto t_\zeta$ from the set of ordinals $\zeta \leq \beta$ (for some suitable β) to K , and to define k_λ by transfinite induction (for $\lambda \leq \beta$) as the union of the k_μ for $\mu < \lambda$ if λ is a limit ordinal, and as $k_\nu(t_\zeta)$ if $\lambda = \nu + 1$, where ζ is the smallest ordinal such that $t_\zeta \notin k_\nu$; γ is then, by definition, the smallest ordinal $\leq \beta$ such that $k_\gamma = K$.

With this in mind, we will define, by transfinite induction, a family of local Noetherian rings A_λ for $\lambda \leq \gamma$, and local homomorphisms $f_{\mu\lambda} : A_\lambda \rightarrow A_\mu$ for $\lambda \leq \mu$, satisfying the following conditions:

- (i) $(A_\lambda, f_{\mu\lambda})$ is an inductive system, and $A_0 = A$;
- (ii) for all λ , we have a k -isomorphism $A_\lambda/\mathfrak{J}A_\lambda \simeq k_\lambda$;
- (iii) for $\lambda \leq \mu$, A_μ is a flat A_λ -module.

So suppose that the A_λ and the $f_{\mu\lambda}$ are defined for $\lambda < \mu < \zeta$, and suppose, first of all, that $\zeta = \zeta + 1$, so that $k_\zeta = k_\zeta(t)$. If t is transcendental over k_ζ , we define A_ζ , following the procedure of (10.3.1.1), to be equal to $(A_\zeta[t])_{\mathfrak{J}A_\zeta[t]}$; the canonical map is $f_{\zeta\zeta}$, and, for $\lambda < \zeta$, we take $f_{\zeta\lambda} = f_{\zeta\zeta} \circ f_{\zeta\lambda}$; the verification of conditions (i) to (iii) is then immediate, given that what we have shown in (10.3.1.1). So now suppose that t is algebraic, and let h be its minimal polynomial in $k_\zeta[T]$, and H a monic polynomial in $A_\zeta[T]$ whose image in $k_\zeta[T]$ is h ; we then take A_ζ to be equal to $A_\zeta[T](H)$, with the $f_{\zeta\lambda}$ being defined as before; the verification of conditions (i) to (iii) then follows from what we have shown in (10.3.1.2).

Now suppose that ζ has no predecessor; we then take A_ζ to be the inductive limit of the inductive system of local rings $(A_\lambda, f_{\mu\lambda})$ for $\lambda < \zeta$; we define $f_{\zeta\lambda}$ as the canonical map for $\lambda < \zeta$. The fact that A_ζ is local and Noetherian, that the $f_{\zeta\lambda}$ are local homomorphism, and that conditions (i) to (iii) are satisfied for $\lambda \leq \zeta$ then follows from the induction hypothesis, and from Lemma (10.3.1.3). With this construction, it is clear that the ring $B = A_\gamma$ satisfies the conditions of (10.3.1). \square

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We note that, by (10.2.1, c), we have a canonical isomorphism

$$(10.3.1.5) \quad \text{gr}(A) \otimes_k K \xrightarrow{\sim} \text{gr}(B).$$

We can also replace B by its $\mathfrak{J}B$ -adic completion \widehat{B} without changing the conclusions of (10.3.1), because \widehat{B} is a flat B -module (0I, 7.3.3), and thus a flat A -module (0I, 6.2.1).

We have also shown the following:

Corollary (10.3.2). — *If K is an extension of finite degree, then we can assume that B is a finite A -algebra.*

§11. SUPPLEMENT ON HOMOLOGICAL ALGEBRA

11.1. Review of spectral sequences

(11.1.1). In the following, we use a more general notion of a spectral sequence than that defined in (T, 2.4); keeping the notations of (T, 2.4), we call a *spectral sequence* in an abelian category \mathcal{C} a system E consisting of the following parts:

- (a) A family (E_r^{pq}) of objects of \mathcal{C} defined for $p, q \in \mathbf{Z}$ and $r \geq 2$.
- (b) A family of morphisms $d_r^{pq} : E_r^{pq} \rightarrow E_r^{p+r, q-r+1}$ such that $d_r^{p+r, q-r+1} d_r^{pq} = 0$. We set $Z_{r+1}(E_r^{pq}) = \text{Ker}(d_r^{pq})$ and $B_{r+1}(E_r^{pq}) = \text{Im}(d_r^{p+r, q-r+1})$, so that

$$B_{r+1}(E_r^{pq}) \subset Z_{r+1}(E_r^{pq}) \subset E_r^{pq}.$$

- (c) A family of isomorphisms $\alpha_r^{pq} : Z_{r+1}(E_r^{pq})/B_{r+1}(E_r^{pq}) \simeq E_{r+1}^{pq}$.

We then define for $k \geq r + 1$, by induction on k , the subobjects $B_k(E_r^{pq})$ and $Z_k(E_r^{pq})$ as the inverse images, under the canonical morphism $E_r^{pq} \rightarrow E_r^{pq}/B_{r+1}(E_r^{pq})$ of the subobjects of this quotient identified via α_r^{pq} with the subobjects $B_k(E_{r+1}^{pq})$ and $Z_k(E_{r+1}^{pq})$ respectively. It is clear that we then have, up to isomorphism,

$$(11.1.1.1) \quad Z_k(E_r^{pq})/B_k(E_r^{pq}) = E_k^{pq} \text{ for } k \geq r + 1,$$

and, if we set $B_r(E_r^{pq}) = 0$ and $Z_r(E_r^{pq}) = E_r^{pq}$, then we have the inclusion relations

$$(11.1.1.2) \quad 0 = B_r(E_r^{pq}) \subset B_{r+1}(E_r^{pq}) \subset B_{r+2}(E_r^{pq}) \subset \cdots \subset Z_{r+2}(E_r^{pq}) \subset Z_{r+1}(E_r^{pq}) \subset Z_r(E_r^{pq}) = E_r^{pq}.$$

The other parts of the data of E are then:

- (d) Two subobjects $B_\infty(E_2^{pq})$ and $Z_\infty(E_2^{pq})$ of E_2^{pq} such that we have $B_\infty(E_2^{pq}) \subset Z_\infty(E_2^{pq})$ and, for every $k \geq 2$,

$$B_k(E_2^{pq}) \subset B_\infty(E_2^{pq}) \text{ and } Z_\infty(E_2^{pq}) \subset Z_k(E_2^{pq}).$$

We set

$$(11.1.1.3) \quad E_\infty^{pq} = Z_\infty(E_2^{pq})/B_\infty(E_2^{pq}).$$

- (e) A family (E^n) of objects of \mathcal{C} , each equipped with a *decreasing filtration* $(F^p(E^n))_{p \in \mathbf{Z}}$. As usual, we denote by $\text{gr}(E^n)$ the graded object associated to the filtered object E^n , the direct sum of the $\text{gr}_p(E^n) = F^p(E^n)/F^{p+1}(E^n)$.

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- (f) For every pair $(p, q) \in \mathbf{Z} \times \mathbf{Z}$, an isomorphism $\beta^{pq} : E_\infty^{pq} \simeq \text{gr}_p(E^{p+q})$.

The family (E^n) , without the filtrations, is called the *abutment* (or *limit*) of the spectral sequence E .

Suppose that the category \mathcal{C} admits infinite direct sums, or that for every $r \geq 2$ and every $n \in \mathbb{Z}$, there are finitely many pairs (p, q) such that $p + q = n$ and $E_r^{pq} \neq 0$ (it suffices for it to hold for $r = 2$). Then the $E_r^{(n)} = \sum_{p+q=n} E_r^{pq}$ are defined, and we if denote by $d_r^{(n)}$ the morphism $E_r^{(n)} \rightarrow E_r^{(n+1)}$ whose restriction to E_r^{pq} is d_r^{pq} for every pair (p, q) such that $p + q = n$, then $d_r^{(n+1)} \circ d_r^{(n)} = 0$, in other words, $(E_r^{(n)})_{n \in \mathbb{Z}}$ is a complex $E_r^{(\bullet)}$ in \mathcal{C} , with differentials of degree $+1$, and it follows from (c) that $H^n(E_r^{(\bullet)})$ is isomorphic to $E_{r+1}^{(n)}$ for every $r \geq 2$.

(11.1.2). A morphism $u : E \rightarrow E'$ from a spectral sequence E to a spectral sequence $E' = (E_r'^{pq}, E'^n)$ consists of systems of morphisms $u_r^{pq} : E_r^{pq} \rightarrow E_r'^{pq}$ and $u^n : E^n \rightarrow E'^n$, the u^n compatible with the filtrations on E^n and E'^n , and the diagrams

$$\begin{array}{ccc} E_r^{pq} & \xrightarrow{d_r^{pq}} & E_r^{p+r, q-r+1} \\ u_r^{pq} \downarrow & & \downarrow u_r^{p+r, q-r+1} \\ E_r'^{pq} & \xrightarrow{d_r'^{pq}} & E_r'^{p+r, q-r+1} \end{array}$$

being commutative; in addition, by passing to quotients, u_r^{pq} gives a morphism $\bar{u}_r^{pq} : Z_{r+1}(E_r^{pq})/B_{r+1}(E_r^{pq}) \rightarrow Z_{r+1}(E_r'^{pq})/B_{r+1}(E_r'^{pq})$ and we must have $\alpha_r'^{pq} \circ \bar{u}_r^{pq} = u_{r+1}^{pq} \circ \alpha_r^{pq}$; finally, we must have $u_2^{pq}(B_\infty(E_2^{pq})) \subset B_\infty(E_2'^{pq})$ and $u_2^{pq}(Z_\infty(E_2^{pq})) \subset Z_\infty(E_2'^{pq})$; by passing to quotients, u_2^{pq} then gives a morphism $u_\infty'^{pq} : E_\infty^{pq} \rightarrow E_\infty'^{pq}$, and the diagram

$$\begin{array}{ccc} E_\infty^{pq} & \xrightarrow{u_\infty'^{pq}} & E_\infty'^{pq} \\ \beta^{pq} \downarrow & & \downarrow \beta'^{pq} \\ \text{gr}_p(E^{p+q}) & \xrightarrow{\text{gr}_p(u^{p+q})} & \text{gr}_p(E'^{p+q}) \end{array}$$

must be commutative.

The above definitions show, by induction on r , that if the u_2^{pq} are isomorphisms, then so are the u_r^{pq} for $r \geq 2$; if in addition we know that $u_2^{pq}(B_\infty(E_2^{pq})) = B_\infty(E_2'^{pq})$ and $u_2^{pq}(Z_\infty(E_2^{pq})) = Z_\infty(E_2'^{pq})$ and the u^n are isomorphisms, then we can conclude that u is an isomorphism.

(11.1.3). Recall that if $(F^p(X))_{p \in \mathbb{Z}}$ is a (decreasing) filtration of an object $X \in \mathcal{C}$, then we say that this filtration is *separated* if $\inf(F^p(X)) = 0$, *discrete* if there exists a p such that $F^p(X) = 0$, *exhaustive* (or *coseparated*) if $\sup(F^p(X)) = X$, *codiscrete* if there exists a p such that $F^p(X) = X$.

We say that a spectral sequence $E = (E_r^{pq}, E^n)$ is *weakly convergent* if we have $B_\infty(E_2^{pq}) = \sup_k(B_k(E_2^{pq}))$ and $Z_\infty(E_2^{pq}) = \inf_k(Z_k(E_2^{pq}))$ (in other words, the objects of $B_\infty(E_2^{pq})$ and $Z_\infty(E_2^{pq})$ are determined from the data of (a) and (c) of the spectral sequence E). We say that the spectral sequence E is *regular* if it is weakly convergent and if in addition:

- (1st) For every pair (p, q) , the decreasing sequence $(Z_k(E_2^{pq}))_{k \geq 2}$ is *stable*; the hypothesis that E is weakly convergent then implies that $Z_\infty(E_2^{pq}) = Z_k(E_2^{pq})$ for k large enough (depending on p and q).
- (2nd) For every n , the filtration $(F^p(E^n))_{p \in \mathbb{Z}}$ of E^n is *discrete* and *exhaustive*.

We say that the spectral sequence E is *coregular* if it is weakly convergent and if in addition:

- (3rd) For every pair (p, q) , the increasing sequence $(B_k(E_2^{pq}))_{k \geq 2}$ is *stable*, which implies that $B_\infty(E_2^{pq}) = B_k(E_2^{pq})$, and as a result, $E_\infty^{pq} = \inf E_k^{pq}$.
- (4th) For every n , the filtration of E^n is *codiscrete*.

Finally, we say that E is *biregular* if it is both regular and coregular, in other words if we have the following conditions:

- (a) For every pair (p, q) , the sequences $(B_k(E_2^{pq}))_{k \geq 2}$ and $(Z_k(E_2^{pq}))_{k \geq 2}$ are *stable* and we have $B_\infty(E_2^{pq}) = B_k(E_2^{pq})$ and $Z_\infty(E_2^{pq}) = Z_k(E_2^{pq})$ for k large enough (which implies that $E_\infty^{pq} = E_k^{pq}$).

(b) For every n , the filtration $(F^p(E^n))_{p \in \mathbb{Z}}$ is *discrete* and *codiscrete* (which we also call *finite*).

The spectral sequences defined in (T, 2.4) are thus biregular spectral sequences.

(11.1.4). Suppose that in the category \mathcal{C} , filtered inductive limits exist and the functor \varinjlim is *exact* (which is equivalent to saying that the axiom (AB 5) of (T, 1.5) is satisfied (cf. T, 1.8)). The condition that the filtration $(F^p(X))_{p \in \mathbb{Z}}$ of an object $X \in \mathcal{C}$ is *exhaustive* is then expressed as $\varinjlim_{p \rightarrow -\infty} F^p(X) = X$. If a spectral sequence E is weakly convergent, then we have $B_\infty(E_2^{pq}) = \varinjlim_{k \rightarrow \infty} B_k(E_2^{pq})$; if in addition $u : E \rightarrow E'$ is a morphism from E to a weakly convergent spectral sequence E' in \mathcal{C} , then we have $u_2^{pq}(B_\infty(E_2^{pq})) = B_\infty(E_2'^{pq})$, by the exactness of \varinjlim . In addition:

Proposition (11.1.5). — *Let \mathcal{C} be an abelian category in which filtered inductive limits are exact, E and E' two regular spectral sequences in \mathcal{C} , $u : E \rightarrow E'$ a morphism of spectral sequences. If the u_2^{pq} are isomorphisms, then so is u .*

Proof. We already know (11.1.2) that the u_r^{pq} are isomorphisms and that

$$u_2^{pq}(B_\infty(E_2^{pq})) = B_\infty(E_2'^{pq});$$

the hypothesis that E and E' are regular also implies that $u_2^{pq}(Z_\infty(E_2^{pq})) = Z_\infty(E_2'^{pq})$, and as u_2^{pq} is an isomorphism, so is u^{pq} ; we thus conclude that $\text{gr}_p(u^{p+q})$ is also an isomorphism. But as the filtrations of the E^n and the E'^n are discrete and exhaustive, this implies that the u^n are also isomorphisms (Bourbaki, *Alg. comm.*, chap. III, §2, n°8, th. 1). \square

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(11.1.6). It follows from (11.1.1.2) and the definition (11.1.1.3) that if, for a spectral sequence E , we have $E_r^{pq} = 0$, then we have $E_k^{pq} = 0$ for $k \geq r$ and $E_\infty^{pq} = 0$. We say that a spectral sequence *degenerates* if there exists an integer $r \geq 2$ and, for every integer $n \in \mathbb{Z}$, an integer $q(n)$ such that $E_r^{n-q, q} = 0$ for every $q \neq q(n)$. We first deduce from the previous remark that we also have $E_k^{n-q, q} = 0$ for $k \geq r$ (including $k = \infty$) and $q \neq q(n)$. In addition, the definition of E_{r+1}^{pq} shows that we have $E_{r+1}^{n-q(n), q(n)} = E_r^{n-q(n), q(n)}$; if E is *weakly convergent*, then we also have $E_\infty^{n-q(n), q(n)} = E_r^{n-q(n), q(n)}$; in other words, for every $n \in \mathbb{Z}$, $\text{gr}_p(E^n) = 0$ for $p \neq q(n)$ and $\text{gr}_{q(n)}(E^n) = E_r^{n-q(n), q(n)}$. If in addition the filtration of E^n is *discrete* and *exhaustive*, then the spectral sequence E is *regular*, and we have $E^n = E_r^{n-q(n), q(n)}$ up to unique isomorphism.

(11.1.7). Suppose that filtered inductive limits exist and are exact in the category \mathcal{C} , and let $(E_\lambda, u_{\mu\lambda})$ be an inductive system (over a filtered set of indices) of spectral sequences in \mathcal{C} . Then the *inductive limit* of this inductive system exists in the additive category of spectral sequences of objects of \mathcal{C} : to see this, it suffices to define $E_r^{pq}, d_r^{pq}, \alpha_r^{pq}, B_\infty(E_2^{pq}), Z_\infty(E_2^{pq}), E^n, F^p(E^n)$, and β^{pq} as the respective inductive limits of the $E_{r,\lambda}^{pq}, d_{r,\lambda}^{pq}, \alpha_{r,\lambda}^{pq}, B_\infty(E_{2,\lambda}^{pq}), Z_\infty(E_{2,\lambda}^{pq}), E_\lambda^n, F^p(E_\lambda^n)$, and β_λ^{pq} ; the verification of the conditions of (11.1.1) follows from the exactness of the functor \varinjlim on \mathcal{C} .

Remark (11.1.8). — Suppose that the category \mathcal{C} is the category of A -modules over a *Noetherian* ring A (resp. a ring A). Then the definitions of (11.1.1) show that if, for a given r , the E_r^{pq} are A -modules of *finite type* (resp. of *finite length*), then so are each of the modules E_s^{pq} for $s \geq r$, hence so is E_∞^{pq} . If in addition the filtration of the abutment/limit (E^n) is *discrete* or *codiscrete* for all n , then we conclude that each of the E^n is also an A -module of *finite type* (resp. of *finite length*).

(11.1.9). We will have to consider conditions which ensure that a spectral sequence E is biregular is a “uniform” way in $p + q = n$. We will then use the following lemma:

Lemma (11.1.10). — *Let (E_r^{pq}) be a family of objects of \mathcal{C} related by the data of (a), (b), and (c) of (11.1.1). For a fixed integer n , the following properties are equivalent:*

- (a) *There exists an integer $r(n)$ such that for $r \geq r(n)$, $p + q = n$ or $p + q = n - 1$, the morphisms d_r^{pq} are zero.*
- (b) *There exists an integer $r(n)$ such that for $p + q = n$ or $p + q = n + 1$, we have $B_r(E_2^{pq}) = B_s(E_2^{pq})$ for $s \geq r \geq r(n)$.*
- (c) *There exists an integer $r(n)$ such that for $p + q = n$ or $p + q = n - 1$, we have $Z_r(E_2^{pq}) = Z_s(E_2^{pq})$ for $s \geq r \geq r(n)$.*

- (d) *There exists an integer $r(n)$ such that for $p + q = n$, we have $B_r(E_2^{pq}) = B_s(E_2^{pq})$ and $Z_r(E_2^{pq}) = Z_s(E_2^{pq})$ for $s \geq r \geq r(n)$.*

Proof. According to the conditions (a), (b), and (c) of (11.1.1), we have that saying $Z_{r+1}(E_2^{pq}) = Z_r(E_2^{pq})$ is equivalent to saying that $d_r^{pq} = 0$ and that saying $B_r(E_2^{p+r, q-r+1}) = B_{r+1}(E_2^{p+r, q-r+1})$ is equivalent to saying that $d_r^{pq} = 0$; the lemma immediately follows from this remark. \square

11.2. The spectral sequence of a filtered complex

(11.2.1). Given an abelian category \mathcal{C} , we will agree to denote by notations such as K^\bullet the *complexes* $(K^i)_{i \in \mathbb{Z}}$ of objects of \mathcal{C} whose differential is of degree $+1$, and by the notations such as K_\bullet the *complexes* $(K_i)_{i \in \mathbb{Z}}$ of objects of \mathcal{C} whose differential is of degree -1 . To each complex $K^\bullet = (K^i)$ whose differential d is of degree $+1$, we can associate a complex $K'_\bullet = (K'_i)$ by setting $K'_i = K^{-i}$, the differential $K'_i \rightarrow K'_{i-1}$ being the operator $d : K^{-i} \rightarrow K^{-i-1}$; and *vice versa*, which, depending on the circumstances, will allow one to consider either one of the types of complexes and translate any result from one type into results for the other. We similarly denote by notations such as $K^{\bullet\bullet} = (K^{ij})$ (resp. $K_{\bullet\bullet} = (K_{ij})$) the *bicomplexes* (or *double complexes*) of objects of \mathcal{C} in which the *two* differentials are of degree $+1$ (resp. -1); we can still pass from one type to the other by changing the signs of the indices, and we have similar notations and remarks for any multicomplexes. The notation K^\bullet and K_\bullet will also be used for **Z**-graded objects of \mathcal{C} , which are not necessarily complexes (they can be considered as such for the zero differentials); for example, we write $H^\bullet(K^\bullet) = (H^i(K^\bullet))_{i \in \mathbb{Z}}$ for the *cohomology* of a complex K^\bullet whose differential is of degree $+1$, and $H_\bullet(K_\bullet) = (H_i(K_\bullet))_{i \in \mathbb{Z}}$ for the *homology* of a complex K_\bullet whose differential is of degree -1 ; when we pass from K^\bullet to K'_\bullet by the method described above, we have $H_i(K'_\bullet) = H^{-i}(K^\bullet)$.

Recall in this case that for a complex K^\bullet (resp. K_\bullet), we will write in general $Z^i(K^\bullet) = \text{Ker}(K^i \rightarrow K^{i+1})$ (“object of cocycles”) and $B^i(K^\bullet) = \text{Im}(K^{i-1} \rightarrow K^i)$ (“object of coboundaries”) (resp. $Z_i(K_\bullet) = \text{Ker}(K_i \rightarrow K_{i-1})$ (“object of cycles”) and $B_i(K_\bullet) = \text{Im}(K_{i+1} \rightarrow K_i)$ (“object of boundaries”)”) so that $H^i(K^\bullet) = Z^i(K^\bullet)/B^i(K^\bullet)$ (resp. $H_i(K_\bullet) = Z_i(K_\bullet)/B_i(K_\bullet)$).

If $K^\bullet = (K^i)$ (resp. $K_\bullet = (K_i)$) is a complex in \mathcal{C} and $T : \mathcal{C} \rightarrow \mathcal{C}'$ a functor from \mathcal{C} to an abelian category \mathcal{C}' , then we denote by $T(K^\bullet)$ (resp. $T(K_\bullet)$) the complex $(T(K^i))$ (resp. $(T(K_i))$) in \mathcal{C}' .

We will not review the definition of the ∂ -functors (T, 2.1), except to note that we *also* say ∂ -functor in place of ∂^* -functor when the morphism ∂ decreases the degree of a unit, the context clarifying this point if there is cause for confusion.

Finally, we say that a *graded object* $(A_i)_{i \in \mathbb{Z}}$ of \mathcal{C} is *bounded below* (resp. *above*) if there exists an i_0 such that $A_i = 0$ for $i < i_0$ (resp. $i > i_0$).

(11.2.2). Let K^\bullet be a complex in \mathcal{C} whose differential d is of degree $+1$, and suppose it is equipped with a *filtration* $F(K^\bullet) = (F^p(K^\bullet))_{p \in \mathbb{Z}}$ consisting of *graded* subobjects of K^\bullet , in other words, $F^p(K^\bullet) = (K^i \cap F^p(K^\bullet))_{i \in \mathbb{Z}}$; in addition, we assume that $d(F^p(K^\bullet)) \subset F^p(K^\bullet)$ for every $p \in \mathbb{Z}$. Let us quickly recall how one *functorially* defines a spectral sequence $E(K^\bullet)$ from K^\bullet (M, XV, 4 and G, I, 4.3). For $r \geq 2$, the canonical morphism $F^p(K^\bullet)/F^{p+r}(K^\bullet) \rightarrow F^p(K^\bullet)/F^{p+1}(K^\bullet)$ defines a morphism in cohomology

$$H^{p+q}(F^p(K^\bullet)/F^{p+r}(K^\bullet)) \longrightarrow H^{p+q}(F^p(K^\bullet)/F^{p+1}(K^\bullet)).$$

We denote by $Z_r^{pq}(K^\bullet)$ the image of this morphism. Similarly, from the exact sequence

$$0 \longrightarrow F^p(K^\bullet)/F^{p+1}(K^\bullet) \longrightarrow F^{p-r+1}(K^\bullet)/F^{p+1}(K^\bullet) \longrightarrow F^{p-r+1}(K^\bullet)/F^p(K^\bullet) \longrightarrow 0,$$

we deduce from the exact sequence in cohomology a morphism

$$H^{p+q-1}(F^{p-r+1}(K^\bullet)/F^p(K^\bullet)) \longrightarrow H^{p+q}(F^p(K^\bullet)/F^{p+1}(K^\bullet)),$$

and we denote by $B_r^{pq}(K^\bullet)$ the image of this morphism; we show that $B_r^{pq}(K^\bullet) \subset Z_r^{pq}(K^\bullet)$ and we take $E_r^{pq}(K^\bullet) = Z_r^{pq}(K^\bullet)/B_r^{pq}(K^\bullet)$; we will not specify the definition of the d_r^{pq} or the α_r^{pq} .

We note here that all the $Z_r^{pq}(K^\bullet)$ and $B_r^{pq}(K^\bullet)$, for p and q fixed, are subobjects of the same object $H^{p+q}(F^p(K^\bullet)/F^{p+1}(K^\bullet))$, which we denote by $Z_1^{pq}(K^\bullet)$; we set $B_1^{pq}(K^\bullet) = 0$, so that the above definitions of $Z_r^{pq}(K^\bullet)$ and $B_r^{pq}(K^\bullet)$ also work for $r = 1$; we still set $E_1^{pq}(K^\bullet) = Z_1^{pq}(K^\bullet)$. We define

d_1^{pq} and α_1^{pq} such that the conditions of (11.1.1) are satisfied for $r = 1$. On the other hand, we define the subobjects $Z_\infty^{pq}(K^\bullet)$ as the image of the morphism

$$H^{p+q}(F^p(K^\bullet)) \longrightarrow H^{p+q}(F^p(K^\bullet)/F^{p+1}(K^\bullet)) = E_1^{pq}(K^\bullet),$$

and $B_\infty^{pq}(K^\bullet)$ as the image of the morphism

$$H^{p+q-1}(K^\bullet/F^p(K^\bullet)) \longrightarrow H^{p+q}(F^p(K^\bullet)/F^{p+1}(K^\bullet)) = E_1^{pq}(K^\bullet),$$

induced as above from the exact sequence in cohomology. We set $Z_\infty(E_2^{pq}(K^\bullet))$ and $B_\infty(E_2^{pq}(K^\bullet))$ to be the canonical images of $E_2^{pq}(K^\bullet)$ in $Z_\infty^{pq}(K^\bullet)$ and $B_\infty^{pq}(K^\bullet)$.

Finally, we denote by $F^p(H^n(K^\bullet))$ the image in $H^n(K^\bullet)$ of the morphism $H^n(F^p(K^\bullet)) \rightarrow H^n(K^\bullet)$ induced from the canonical injection $F^p(K^\bullet) \rightarrow K^\bullet$; by the exact sequence in cohomology, this is also the kernel of the morphism $H^n(K^\bullet) \rightarrow H^n(K^\bullet/F^p(K^\bullet))$. This defines a filtration on $E^n(K^\bullet) = H^n(K^\bullet)$; we will not give here the definition of the isomorphisms β^{pq} .

(11.2.3). The *functorial* nature of $E(K^\bullet)$ is understood in the following way: given two *filtered* complexes K^\bullet and K'^\bullet in \mathcal{C} and a morphism of complexes $u : K^\bullet \rightarrow K'^\bullet$ that is *compatible with the filtrations*, we induce in an evident way the morphisms u_r^{pq} (for $r \geq 1$) and u^n , and we show that these morphisms are compatible with the d_r^{pq} , α_r^{pq} , and β^{pq} in the sense of (11.1.2), and thus given a well-defined morphism $E(u) : E(K^\bullet) \rightarrow E(K'^\bullet)$ of spectral sequences. In addition, we show that if u and v are morphisms $K^\bullet \rightarrow K'^\bullet$ of the above type, *homotopic in degree $\leq k$* , then $u_r^{pq} = v_r^{pq}$ for $r > k$ and $u^n = v^n$ for all n (M, XV, 3.1).

(11.2.4). Suppose that filtered inductive limits in \mathcal{C} are exact. Then if the filtration $(F^p(K^\bullet))$ of K^\bullet is *exhaustive*, then so is the filtration $(F^p(H^n(K^\bullet)))$ for all n , since by hypothesis we have $K^\bullet = \varinjlim_{p \rightarrow -\infty} F^p(K^\bullet)$ and since the hypothesis on \mathcal{C} implies that cohomology commutes with inductive limits. In addition, for the same reason, we have $B_\infty(E_2^{pq}(K^\bullet)) = \sup_k B_k(E_2^{pq}(K^\bullet))$. We say that the filtration $(F^p(K^\bullet))$ of K^\bullet is *regular* if for every n there exists an integer $u(n)$ such that $H^n(F^p(K^\bullet)) = 0$ for $p > u(n)$. This is particularly the case when the filtration of K^\bullet is *discrete*. When the filtration of K^\bullet is regular and exhaustive, and filtered inductive limits are exact in \mathcal{C} , we have (M, XV, 4) that the spectral sequence $E(K^\bullet)$ is *regular*.

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11.3. The spectral sequences of a bicomplex

(11.3.1). With regard the conventions for bicomplexes, we follow those of (T, 2.4) rather than those of (M), the two differentials d' and d'' (of degree $+1$) of such a bicomplex $K^{\bullet\bullet} = (K^{ij})$ being thus assumed to be *permutable*. Suppose that *one* of the following two conditions is satisfied: 1st. *Infinite direct sums* exist in \mathcal{C} ; 2nd. For all $n \in \mathbf{Z}$, there is only a *finite* number of pairs (p, q) such that $p + q = n$ and $K^{pq} \neq 0$. Then, the bicomplex $K^{\bullet\bullet}$ defines a (simple) *complex* $(K'^n)_{n \in \mathbf{Z}}$ with $K'^n = \sum_{i+j=n} K^{ij}$, the differential d (of degree $+1$) of this complex being given by $dx = d'x + (-1)^i d''x$ for $x \in K^{ij}$. When we later speak of the spectral sequence of a (simple) *complex* that is *defined by a bicomplex* $K^{\bullet\bullet}$, it will always be understood that of the above conditions is satisfied. We adopt the analogous conventions for multicomplexes.

We denote by $K^{i,\bullet}$ (resp. $K^{\bullet,j}$) the simple complex $(K^{ij})_{j \in \mathbf{Z}}$ (resp. $(K^{ij})_{i \in \mathbf{Z}}$), by $Z_{\Pi}^p(K^{i,\bullet})$, $B_{\Pi}^p(K^{i,\bullet})$, $H_{\Pi}^p(K^{i,\bullet})$ (resp. $Z_{\Pi}^p(K^{\bullet,j})$, $B_{\Pi}^p(K^{\bullet,j})$, $H_{\Pi}^p(K^{\bullet,j})$) its p objects of cocycles, of coboundaries, and of cohomology, respectively; the differential $d' : K^{i,\bullet} \rightarrow K^{i+1,\bullet}$ is a morphism of complexes, which thus gives an operator on the cocycles, coboundaries, and cohomology,

$$\begin{aligned} d' : Z_{\Pi}^p(K^{i,\bullet}) &\longrightarrow Z_{\Pi}^p(K^{i+1,\bullet}), \\ d' : B_{\Pi}^p(K^{i,\bullet}) &\longrightarrow B_{\Pi}^p(K^{i+1,\bullet}), \\ d' : H_{\Pi}^p(K^{i,\bullet}) &\longrightarrow H_{\Pi}^p(K^{i+1,\bullet}), \end{aligned}$$

and it is clear that for these operators, $(Z_{\Pi}^p(K^{i,\bullet}))_{i \in \mathbf{Z}}$, $(B_{\Pi}^p(K^{i,\bullet}))_{i \in \mathbf{Z}}$, and $(H_{\Pi}^p(K^{i,\bullet}))_{i \in \mathbf{Z}}$ are complexes; we denote the complex $(H_{\Pi}^p(K^{i,\bullet}))_{i \in \mathbf{Z}}$ by $H_{\Pi}^p(K^{\bullet\bullet})$, its q objects of cocycles, coboundaries, and cohomology by $Z_{\Pi}^q(H_{\Pi}^p(K^{\bullet\bullet}))$, $B_{\Pi}^q(H_{\Pi}^p(K^{\bullet\bullet}))$, and $H_{\Pi}^q(H_{\Pi}^p(K^{\bullet\bullet}))$. We similarly define the complexes $H_{\Pi}^p(K^{\bullet\bullet})$ and their cohomology objects $H_{\Pi}^q(H_{\Pi}^p(K^{\bullet\bullet}))$. Recall on the other hand that $H^n(K^{\bullet\bullet})$ denotes the n object of the cohomology of the (simple) complex defined by $K^{\bullet\bullet}$.

(11.3.2). On the complex defined by a bicomplex $K^{\bullet\bullet}$, we can consider two canonical filtrations $(F_I^p(K^{\bullet\bullet}))$ and $(F_{II}^p(K^{\bullet\bullet}))$ given by

$$(11.3.2.1) \quad F_I^p(K^{\bullet\bullet}) = \left(\sum_{i+j=n, i \geq p} K^{ij} \right)_{n \in \mathbb{Z}} \quad \text{and} \quad F_{II}^p(K^{\bullet\bullet}) = \left(\sum_{i+j=n, j \geq p} K^{ij} \right)_{n \in \mathbb{Z}},$$

which, by definition, are graded subobjects of the (simple) complex defined by $K^{\bullet\bullet}$, and thus make this complex a filtered complex; moreover, it is clear that these filtrations are *exhaustive* and *separated*.

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There corresponds to each of these filtrations a spectral sequence (11.2.2); we denote by $'E(K^{\bullet\bullet})$ and $''E(K^{\bullet\bullet})$ the spectral sequences corresponding to $(F_I^p(K^{\bullet\bullet}))$ and $(F_{II}^p(K^{\bullet\bullet}))$ respectively, called the *spectral sequence of the bicomplex $K^{\bullet\bullet}$* , and both having as their abutment the cohomology $(H^n(K^{\bullet\bullet}))$. We show in addition (M, XV, 6) that we have

$$(11.3.2.2) \quad 'E_2^{pq}(K^{\bullet\bullet}) = H_1^p(H_{II}^q(K^{\bullet\bullet})), \quad ''E_2^{pq}(K^{\bullet\bullet}) = H_{II}^p(H_1^q(K^{\bullet\bullet})).$$

Every morphism $u : K^{\bullet\bullet} \rightarrow K'^{\bullet\bullet}$ of bicomplexes is *ipso facto* compatible with the filtrations of the same type of $K^{\bullet\bullet}$ and $K'^{\bullet\bullet}$, thus define a morphism for each of the two spectral sequences; in addition, two *homotopic* morphisms define a homotopy of order ≤ 1 of the corresponding filtered (simple) complexes, thus the *same* morphism for each of the two spectral sequences (M, XV, 6.1).

Proposition (11.3.3). — *Let $K^{\bullet\bullet} = (K^{ij})$ be a bicomplex in an abelian category \mathcal{C} .*

- (i) *If there exist i_0 and j_0 such that $K^{ij} = 0$ for $i < i_0$ or $j < j_0$ (resp. $i > i_0$ or $j > j_0$), then the two spectral sequences $'E(K^{\bullet\bullet})$ and $''E(K^{\bullet\bullet})$ are biregular.*
- (ii) *If there exist i_0 and i_1 such that $K^{ij} = 0$ for $i < i_0$ or $i > i_1$ (resp. if there exist j_0 and j_1 such that $K^{ij} = 0$ for $j < j_0$ or $j > j_1$), then the two spectral sequences $'E(K^{\bullet\bullet})$ and $''E(K^{\bullet\bullet})$ are biregular.*
- (iii) *If there exists i_0 such that $K^{ij} = 0$ for $i > i_0$ (resp. if there exists j_0 such that $K^{ij} = 0$ for $j < j_0$), then the spectral sequence $'E(K^{\bullet\bullet})$ is regular.*
- (iv) *If there exists i_0 such that $K^{ij} = 0$ for $i < i_0$ (resp. if there exists j_0 such that $K^{ij} = 0$ for $j > j_0$), then the spectral sequence $''E(K^{\bullet\bullet})$ is regular.*

Proof. The proposition follows immediately from the definitions (11.1.3) and from (11.2.4), □

§12. SUPPLEMENT ON SHEAF COHOMOLOGY

12.1. Cohomology of sheaves of modules on ringed spaces

§13. PROJECTIVE LIMITS IN HOMOLOGICAL ALGEBRA

§14. COMBINATORIAL DIMENSION OF A TOPOLOGICAL SPACE

Summary

- §14. Combinatorial dimension of a topological space.
- §15. M -regular sequences and \mathcal{F} -regular sequences.
- §16. Dimension and depth of Noetherian local rings.
- §17. Regular rings.
- §18. Supplement on extensions of algebras.
- §19. Formally smooth algebras and Cohen rings.
- §20. Derivations and differentials.
- §21. Differentials in rings of characteristic p .
- §22. Differential criteria for smoothness and regularity.
- §23. Japanese rings.

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Almost all of the preceding sections have been focused on the exposition of ideas of commutative algebra that will be used throughout Chapter IV. Even though a large amount of these ideas already appear in multiple works ([CC], [Sam53a], [SZ60], [Ser55], [Nag62]), we thought that it would be more practical for the reader to have a coherent, vaguely independent exposition. Together with

§§5, 6, and 7 of Chapter IV (where we use the language of schemes), these sections constitute, in the middle of our treatise, a miniature special treatise, somewhat independent of Chapters I to III, and one that aims to present, in a coherent manner, the properties of rings that “behave well” with respect to operations such as completion, or integral closure, by systematically associating these properties to more general ideas.¹¹

14.1. Combinatorial dimension of a topological space

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(14.1.1). Let I be an ordered set; a *chain* of elements of I is, by definition, a strictly-increasing finite sequence $i_0 < i_1 < \dots < i_n$ of elements of I ($n \geq 0$); by definition, the *length* of this chain is n . If X is a topological space, the set of its *irreducible closed* subsets is ordered by inclusion, and so we have the notion of a *chain* of irreducible closed subsets of X .

Definition (14.1.2). — Let X be a topological space. We define the combinatorial dimension of X (or simply the dimension of X , if there is no risk of confusion), denoted by $\dim c(X)$ (or simply $\dim(X)$), to be the upper bound of lengths of chains of irreducible closed subsets of X . For all $x \in X$, we define the combinatorial dimension of X at x (or simply the dimension of X at x), denoted by $\dim_x(X)$, to be the number $\inf_U(\dim(U))$, where U varies over the open neighbourhoods of x in X .

It follows from this definition that we have

$$\dim(\emptyset) = -\infty$$

(the upper bound in $\bar{\mathbf{R}}$ of the empty set being $-\infty$). If (X_α) is the family of irreducible components of X , then we have

$$(14.1.2.1) \quad \dim(X) = \sup_\alpha \dim(X_\alpha),$$

because every chain of irreducible closed subsets of X is, by definition, contained in some irreducible component of X , and, conversely, the irreducible components are closed in X , so every irreducible closed subset of an X_α is a irreducible closed subset of X .

Definition (14.1.3). — We say that a topological space X is equidimensional if all its irreducible components have the same dimension (which is thus equal to $\dim(X)$, by (14.1.2.1)).

Proposition (14.1.4). —

- (i) For every closed subset Y of a topological space X , we have $\dim(Y) \leq \dim(X)$.
- (ii) If a topological space X is a finite union of closed subsets X_i , then we have $\dim(X) = \sup_i \dim(X_i)$.

Proof. For every irreducible closed subset Z of Y , the closure \bar{Z} of Z in X is irreducible (0I, 2.1.2), and $\bar{Z} \cap Y = Z$, whence (i). Now, if $X = \bigcup_{i=1}^n X_i$, where the X_i are closed, then every irreducible closed subset of X is contained in one of the X_i (0I, 2.1.1), and so every chain of irreducible closed subsets of X is contained in one of the X_i , whence (ii). \square

From (14.1.4, i), we see that, for all $x \in X$, we can also write

$$(14.1.4.1) \quad \dim_x(X) = \lim_U \dim(U),$$

where the limit is taken over the downward-directed set of open neighbourhoods of x in X .

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Corollary (14.1.5). — Let X be a topological space, x a point of X , U a neighbourhood of x , and Y_i ($1 \leq i \leq n$) closed subsets of U such that, for all i , $x \in Y_i$, and such that U is the union of the Y_i . Then we have

$$(14.1.5.1) \quad \dim_x(X) = \sup_i (\dim_x(Y_i)).$$

¹¹The majority of properties which we discuss were discovered by Chevalley, Zariski, Nagata, and Serre. The method used here was first developed in the autumn of 1961, in a course taught at Harvard University by A. Grothendieck.

Proof. It follows from (14.1.4, ii) that we have $\dim_x(X) = \inf_V(\sup_i(\dim(Y_i \cap V)))$, where V ranges over the set of open neighbourhoods of x that are contained in U ; similarly, we have $\dim_x(Y_i) = \inf_V(\dim(Y_i \cap V))$ for all i . The corollary is thus evident if

$$\sup_i(\dim_x(Y_i)) = +\infty;$$

if this were not the case, then there would be an open neighbourhood $V_0 \subset U$ of x such that $\dim(Y_i \cap V) = \dim_x(Y_i)$ for $1 \leq i \leq n$ and for all $V \subset V_0$, whence the conclusion. \square

Proposition (14.1.6). — *For every topological space X , we have $\dim(X) = \sup_{x \in X} \dim_x(X)$.*

Proof. It follows from Definition (14.1.2) and Proposition (14.1.4) that $\dim_x(X) \leq \dim(X)$ for all $x \in X$. Now, let $Z_0 \subset Z_1 \subset \dots \subset Z_n$ be a chain of irreducible closed subsets of X , and let $x \in Z_0$; for every open subset $U \subset X$ that contains x , $U \cap Z_i$ is irreducible (0_I, 2.1.6) and closed in U , and since we have $\overline{U \cap Z_i} = Z_i$ in X , the $U \cap Z_i$ are pairwise distinct; thus $\dim(U) \geq n$, which finishes the proof. \square

Corollary (14.1.7). — *If (X_α) is an open, or closed and locally finite, cover of X , then $\dim(X) = \sup_\alpha(\dim(X_\alpha))$.*

Proof. If X_α is a neighbourhood of $x \in X$, then $\dim_x(X) \leq \dim(X_\alpha)$, whence the claim for open covers. On the other hand, if the X_α are closed, and U is a neighbourhood of $x \in X$ which meets only finitely many of the X_α , then

$$\dim_x(X) \leq \dim(U) = \sup_\alpha(\dim(U \cap X_\alpha)) \leq \sup_\alpha(\dim(X_\alpha))$$

by (14.1.4), whence the other claim. \square

Corollary (14.1.8). — *Let X be a Noetherian Kolmogoroff space (0_I, 2.1.3), and F the set of closed points of X . Then $\dim(X) = \sup_{x \in F} \dim_x(X)$.*

Proof. With the notation from the proof of (14.1.6), it suffices to note that there exists a closed point in Z_0 (0_I, 2.1.3). \square

Proposition (14.1.9). — *Let X be a nonempty Noetherian Kolmogoroff space. To have $\dim(X) = 0$, it is necessary and sufficient for X to be finite and discrete.*

Proof. If a space X is separated (and *a fortiori* if X is a discrete space), then all the irreducible closed subsets of X are single points, and so $\dim(X) = 0$. Conversely, suppose that X is a Noetherian Kolmogoroff space such that $\dim(X) = 0$; since every irreducible component of X contains a closed point (0_I, 2.1.3), it must be exactly this single point. Since X has only a finite number of irreducible components, it is thus finite and discrete. \square

Corollary (14.1.10). — *Let X be a Noetherian Kolmogoroff space. For a point $x \in X$ to be isolated, it is necessary and sufficient to have $\dim_x(X) = 0$.*

Proof. The condition is clearly necessary (without any hypotheses on X). It is also sufficient, because it implies that $\dim(U) = 0$ for any open neighbourhood U of x , and since U is a Noetherian Kolmogoroff space, U is finite and discrete. \square

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Proposition (14.1.11). — *The function $x \mapsto \dim_x(X)$ is upper semi-continuous on X .*

Proof. It is clear that this function is upper semi-continuous at every point where its value is $+\infty$. So suppose that $\dim_x(X) = n < +\infty$; then Equation (14.1.4.1) shows that there exists an open neighbourhood U_0 of x such that $\dim(U) = n$ for every open neighbourhood $U \subset U_0$ of x . So, for all $y \in U_0$ and every open neighbourhood $V \subset U_0$ of y , we have $\dim(V) \leq \dim(U_0) = n$ (14.1.4); we thus deduce from (14.1.4.1) that $\dim_y(X) \leq n$. \square

Remark (14.1.12). — *If X and Y are topological spaces, and $f : X \rightarrow Y$ a continuous map, then it can be the case that $\dim(f(X)) > \dim(X)$; we obtain such an example by taking X to be a discrete space with 2 points, a and b , and Y to be the set $\{a, b\}$ endowed with the topology for which¹²*

¹²[Trans.] This is now often referred to as the Sierpiński space, or the connected two-point set.

the closed sets are \emptyset , $\{a\}$, and $\{a, b\}$; if $f : X \rightarrow Y$ is the identity map, then $\dim(Y) = 1$ and $\dim(X) = 0$. We note that Y is the spectrum of a discrete valuation ring A , of which a is the unique closed point, and b the generic point; if K and k are the field of fractions and the residue field of A (respectively), then X is the spectrum of the ring $k \times K$, and f is the continuous map corresponding to the homomorphism $(\phi, \psi) : A \rightarrow k \times K$, where $\phi : A \rightarrow k$ and $\psi : A \rightarrow K$ are the canonical homomorphism (cf. (IV, 5.4.3)).

14.2. Codimension of a closed subset

Definition (14.2.1). — Given an irreducible closed subset Y of a topological space X , we define the combinatorial codimension (or simply codimension) of Y in X , denoted by $\text{codim}(Y, X)$, to be the upper bound of the lengths of chains of irreducible closed subsets of X of which Y is the smallest element. If Y is an arbitrary closed subset of X , then we define the codimension of Y in X , again denoted by $\text{codim}(Y, X)$, to be the lower bound of the codimensions in X of the irreducible components of Y . We say that X is equicodimensional if all the minimal irreducible closed subsets of X has the same codimension in X .

It follows from this definition that $\text{codim}(\emptyset, X) = +\infty$, since the lower bound of the empty set of $\bar{\mathbf{R}}$ is $+\infty$. If Y is closed in X , and if (X_α) (resp. (Y_α)) is the family of irreducible components of X (resp. Y), then every Y_β is contained in some X_α , and, more generally, every chain of irreducible closed subsets of X of which Y_β is the smallest element is formed of subsets of some X_α ; we thus have

$$(14.2.1.1) \quad \text{codim}(Y, X) = \inf_{\beta} (\sup_{\alpha} (\text{codim}(Y_{\beta}, X_{\alpha}))),$$

where, for every β, α ranges over the set of indices such that $Y_{\beta} \subset X_{\alpha}$.

Proposition (14.2.2). — *Let X be a topological space.*

(i) *If Φ is the set of irreducible closed subsets of X , then*

$$(14.2.2.1) \quad \dim(X) = \sup_{Y \in \Phi} (\text{codim}(Y, X)).$$

(ii) *For every nonempty closed subset Y of X , we have*

$$(14.2.2.2) \quad \dim(Y) + \text{codim}(Y, X) \leq \dim(X).$$

(iii) *If Y, Z , and T are closed subsets of X such that $Y \subset Z \subset T$, then*

$$(14.2.2.3) \quad \text{codim}(Y, Z) + \text{codim}(Z, T) \leq \text{codim}(Y, T).$$

(iv) *For a closed subset Y of X to be such that $\text{codim}(Y, X) = 0$, it is necessary and sufficient for Y to contain an irreducible component of X .*

Proof. Claims (i) and (iv) are immediate consequences of Definition (14.2.1). To show (ii), we can restrict to the case where Y is irreducible, and then the equation follows from Definitions (14.1.1) and (14.2.1). Finally, to show (iii), we can, by Definition (14.2.1), first restrict to the case where Y is irreducible; then $\text{codim}(Y, Z) = \sup_{\alpha} (\text{codim}(Y, Z_{\alpha}))$ for the irreducible components Z_{α} of Z that contain Y ; it is clear that $\text{codim}(Y, T) \geq \text{codim}(Y, Z)$, so the inequality is true if $\text{codim}(Y, Z) = +\infty$; but if this were not the case, then there would exist some α such that $\text{codim}(Y, Z) = \text{codim}(Y, Z_{\alpha})$, and by (14.2.1), we can restrict to the case where Z itself is irreducible; but then the inequality in (14.2.2.3) is an evident consequence of Definition (14.2.1). \square

Proposition (14.2.3). — *Let X be a topological space, and Y a closed subset of X . For every open subset U of X , we have*

$$(14.2.3.1) \quad \text{codim}(Y \cap U, U) \geq \text{codim}(Y, X).$$

Furthermore, for this inequality (14.2.3.1) to be an equality, it is necessary and sufficient to have $\text{codim}(Y, X) = \inf_{\alpha} (\text{codim}(Y_{\alpha}, X))$, where (Y_{α}) is the family of irreducible components of Y that meet U .

Proof. We know (0_I, 2.1.6) that $Z \mapsto \overline{Z}$ is a bijection from the set of irreducible closed subsets of U to the set of irreducible closed subsets of X that meet U , and, in particular, induces a correspondence between the irreducible components of $Y \cap U$ and the irreducible components of Y that meet U ; if Y_α is one of the latter such components, then we have $\text{codim}(Y_\alpha, X) = \text{codim}(Y_\alpha \cap U, U)$, and the proposition then follows from Definition (14.2.1). \square

Definition (14.2.4). — Let X be a topological space, Y a closed subset of X , and x a point of X . We define the codimension of Y in X at the point x , denoted by $\text{codim}_x(Y, X)$, to be the number $\sup_U(\text{codim}(Y \cap U, U))$, where U ranges over the set of open neighbourhoods of x in X .

By (14.2.3), we can also write

$$(14.2.4.1) \quad \text{codim}_x(Y, X) = \lim_U(\text{codim}(Y \cap U, U)),$$

where the limit is taken over the downward-directed set of open neighbourhoods of x in X . We note that we have

$$\text{codim}_x(Y, X) = +\infty \text{ if } x \in X - Y.$$

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Proposition (14.2.5). — If $(Y_i)_{1 \leq i \leq n}$ is a finite family of closed subsets of a topological space X , and Y is the union of this family, then

$$(14.2.5.1) \quad \text{codim}(Y, X) = \inf_i(\text{codim}(Y_i, X)).$$

Proof. Every irreducible component of one of the Y_i is contained in an irreducible component of Y , and, conversely, every irreducible component of Y is also an irreducible component of one of the Y_i (0_I, 2.1.1); the conclusion then follows from Definition (14.2.1) and the inequality in (14.2.2.3). \square

Corollary. — Let X be a topological space, and Y a locally-Noetherian closed subspace of X .

- (i) For all $x \in X$, there exists only a finite number of irreducible components Y_i ($1 \leq i \leq n$) of Y that contain x , and we have $\text{codim}_x(Y, X) = \inf_i(\text{codim}(Y_i, X))$.
- (ii) The function $x \mapsto \text{codim}_x(Y, X)$ is lower semi-continuous on X .

Proof. By hypothesis, there exists an open neighbourhood U_0 of x in X such that $Y \cap U_0$ is Noetherian, and so U_0 has only a finite number of irreducible components, which are the intersections of U_0 with the irreducible components of Y ; a fortiori there are only a finite number of irreducible components Y_i ($1 \leq i \leq n$) of Y that contain x , and we can, by replacing U_0 with an open neighbourhood $U \subset U_0$ of x that doesn't meet any of the Y_j that do not contain x , assume that the $Y_i \cap U$ are the irreducible components of $Y \cap U$; for every open neighbourhood $V \subset U$ of x in X , the $Y_i \cap V$ are thus the irreducible components of $Y \cap V$, and (14.2.3) then shows that $\text{codim}(Y_i, X) = \text{codim}(Y_i \cap V, V)$, which proves (i). Further, for every $x' \in U$, the irreducible components of Y that contain x' are certain Y_i , and so $\text{codim}_{x'}(Y, X) \geq \text{codim}_x(Y, X)$, which proves (ii). \square

14.3. The chain condition

(14.3.1). In a topological space X , we say that a chain $Z_0 \subset Z_1 \subset \dots \subset Z_n$ of irreducible closed subsets is *saturated* if there does not exist an irreducible closed subset Z' , distinct from each of the Z_i , such that $Z_k \subset Z' \subset Z_{k+1}$ for any k .

Proposition (14.3.2). — Let X be a topological space such that, for any two irreducible closed subsets Y and Z of X with $Y \subset Z$, we have $\text{codim}(Y, Z) < +\infty$. The following two conditions are equivalent.

- (a) Any two saturated chains of closed irreducible subsets of X that have the same first and last elements as one another have the same length.
- (b) If Y, Z , and T are irreducible closed subsets of X such that $Y \subset Z \subset T$, then

$$(14.3.2.1) \quad \text{codim}(Y, T) = \text{codim}(Y, Z) + \text{codim}(Z, T).$$

Proof. It is immediate that (a) implies (b). Conversely, suppose that (b) is satisfied, and we will show that if we have two saturated chains with the same first and last elements as one another, of lengths m and $n \leq m$ (respectively), then $m = n$. We proceed by induction on n , with the proposition being clear for $n = 1$. So suppose that $1 < n < m$, and let $Z_0 \subset Z_1 \subset \dots \subset Z_n$ be a

saturated chain such that there exists another saturated chain, with first element Z_0 and last element Z_n , of length m . Since $\text{codim}(Z_0, Z_n) \geq m > n$, and $\text{codim}(Z_0, Z_1) = 1$, it follows from (b) that $\text{codim}(Z_1, Z_n) = \text{codim}(Z_0, Z_n) - 1 > n - 1$, which contradicts our induction hypothesis. \square

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When the conditions of (14.3.2) are satisfied, we say that X satisfies the *chain condition*, or that it is a *catenary space*. It is clear that every closed subspace of a catenary space is catenary.

Proposition (14.3.3). — *Let X be a Noetherian Kolmogoroff space of finite dimension. The following conditions are equivalent.*

- (a) Any two maximal chains of irreducible closed subsets of X have the same length.
- (b) X is equidimensional, equicodimensional, and catenary.
- (c) X is equidimensional, and, for any irreducible closed subsets Y and Z of X with $Y \subset Z$, we have

$$(14.3.3.1) \quad \dim(Z) = \dim(Y) + \text{codim}(Y, Z).$$

- (d) X is equicodimensional, and, for any irreducible closed subsets Y and Z of X with $Y \subset Z$, we have

$$(14.3.3.2) \quad \text{codim}(Y, Z) = \text{codim}(Y, X) + \text{codim}(Z, X).$$

Proof. The hypotheses on X imply that the first and last elements of a maximal chain of irreducible closed subsets of X are necessarily a closed point and an irreducible component of X (respectively (0I, 2.1.3)); further, every saturated chain with first element Y and last element Z (thus $Y \subset Z$) is contained in a maximal chain whose elements differ from those of the given chain, or are contained in Y , or contain Z . These remarks immediately establish the equivalence between (a) and (b), and also show that if (a) is satisfied, then we have, for every irreducible closed subset Y to X ,

$$(14.3.3.3) \quad \dim(Y) + \text{codim}(Y, X) = \dim(X);$$

from (14.3.2.1), we immediately deduce (14.3.3.1) and (14.3.3.2) from (14.3.3.3). Conversely, (14.3.3.1) implies (14.3.2.1), and so (14.3.3.1) implies the chain condition, by (14.3.2); further, by applying (14.3.3.1) to the case where Y is a single closed point x of X , and Z is an irreducible component of X , we get that $\text{codim}(\{x\}, X) = \dim(Z)$; we thus conclude that (c) implies (b). Similarly, (14.3.3.2) implies (14.3.2.1), and thus the chain condition; further, with the same choice of Y and Z as above, (14.3.3.2) again implies that $\text{codim}(\{x\}, X) = \dim(Z)$, and so (since every irreducible component of X contains a closed point, by (0I, 2.1.3)), (d) implies (b). \square

We say that a Noetherian Kolmogoroff space is *biequidimensional* if it is of finite dimension and if it verifies any of the (equivalent) conditions of (14.3.3).

Corollary (14.3.4). — *Let X be a biequidimensional Noetherian Kolmogoroff space; then, for every closed point x of X , and every irreducible component Z of X , we have*

$$(14.3.4.1) \quad \dim(X) = \dim(Z) = \text{codim}(\{x\}, X) = \dim_x(X).$$

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Proof. The last equality follows from the fact that, if $Y_0 = \{x\} \subset Y_1 \subset \dots \subset Y_m$ is a maximal chain of irreducible closed subsets of X , and U an open neighbourhood of x , then the $U \cap Y_i$ are pairwise disjoint irreducible closed subsets of U (because $\overline{U \cap Y_i} = Y_i$), whence $\dim(U) = \dim(X)$, by (14.1.4). \square

Corollary (14.3.5). — *Let X be a Noetherian Kolmogoroff space; if X is biequidimensional, then so is every union of irreducible components of X , and every irreducible closed subset of X . In addition, for every closed subset Y of X , we have*

$$(14.3.5.1) \quad \dim(Y) + \text{codim}(Y, X) = \dim(X).$$

Proof. Every chain of irreducible closed subsets of X is contained in an irreducible component of X , and so the first claim follows immediately from (14.3.3). Further, if X' is an irreducible closed subset of X , then X' trivially satisfies the conditions of (14.3.3, c), whence the second claim.

Finally, to show (14.3.5.1), note that we have seen, in the proof of (14.3.3), that this equation is true whenever Y is irreducible; if Y_i ($1 \leq i \leq m$) are the irreducible components of Y , then the Y_i for which $\dim(Y_i)$ is the largest are also those for which $\text{codim}(Y_i, X)$ is the smallest; so (14.3.5.1) follows from the definitions of $\dim(Y)$ and $\text{codim}(Y, X)$. \square

Remark (14.3.6). — The reader will note that the proof of (14.3.2) applies to any ordered set, and the fact that we are working with the example of a set of irreducible closed subsets of a topological space is not used at all in the proof. It is the same in the proof of (14.3.3), which holds, more generally, for any ordered set E such that, for all $x \in E$, there exists some $z \leq x$ which is *minimal* in E , and such that the length of chains of elements of E is bounded.

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